“THE GREAT DEBATE”: UNUM ARITHMETIC POSITION STATEMENT

Prof. John L. Gustafson

A*STAR-CRC and National University of Singapore

July 12, 2016

ARITH23, Santa Clara California

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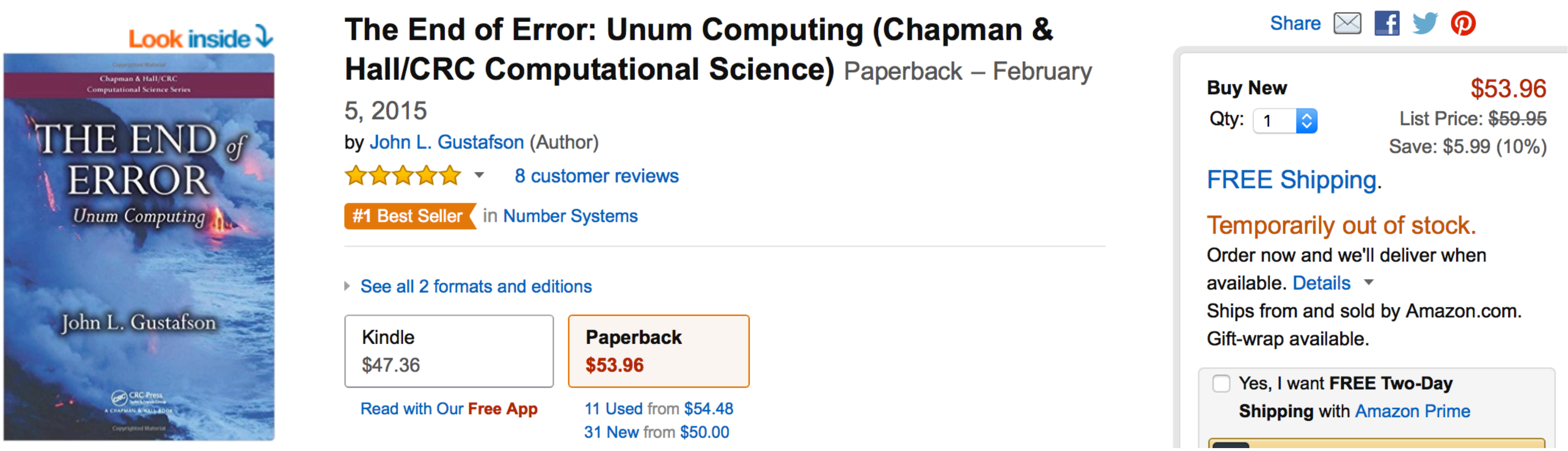
- Then this happened (Amazon.com):



The End of Error: Unum Computing (Chapman & Hall/CRC Computational Science) Paperback – February 5, 2015

by John L. Gustafson (Author)

★★★★★ 8 customer reviews

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The Wrath of Kahan: A Bitter Blog

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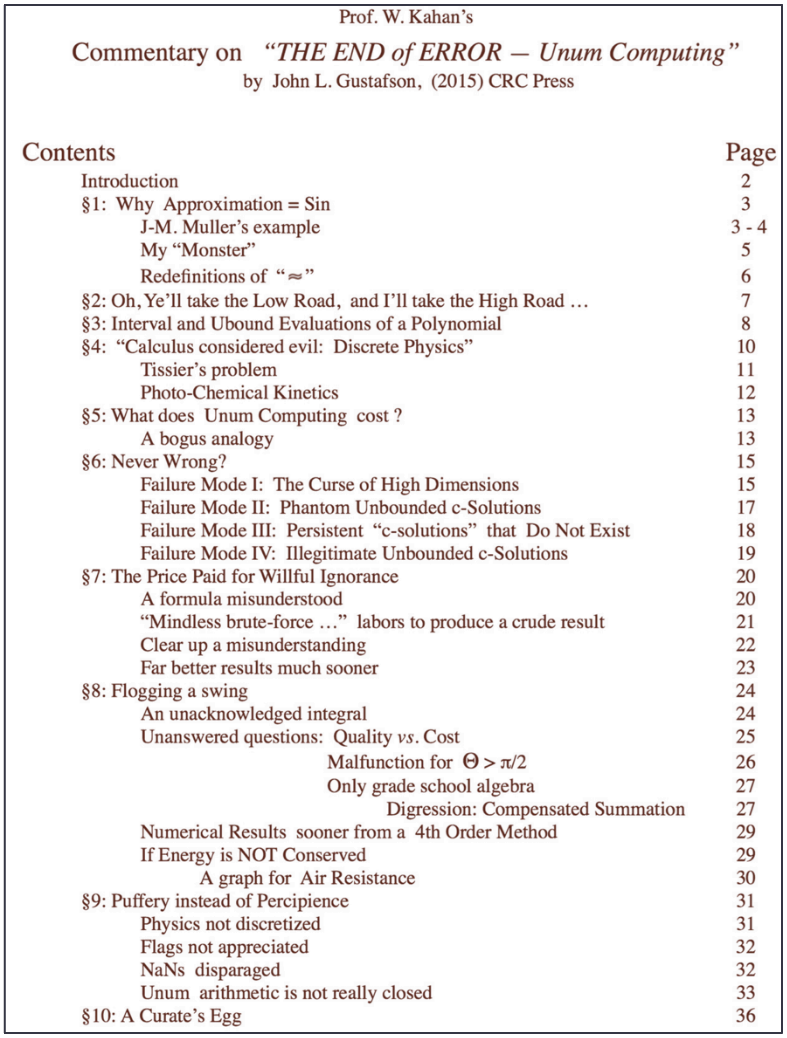

Prof. W. Kahan's

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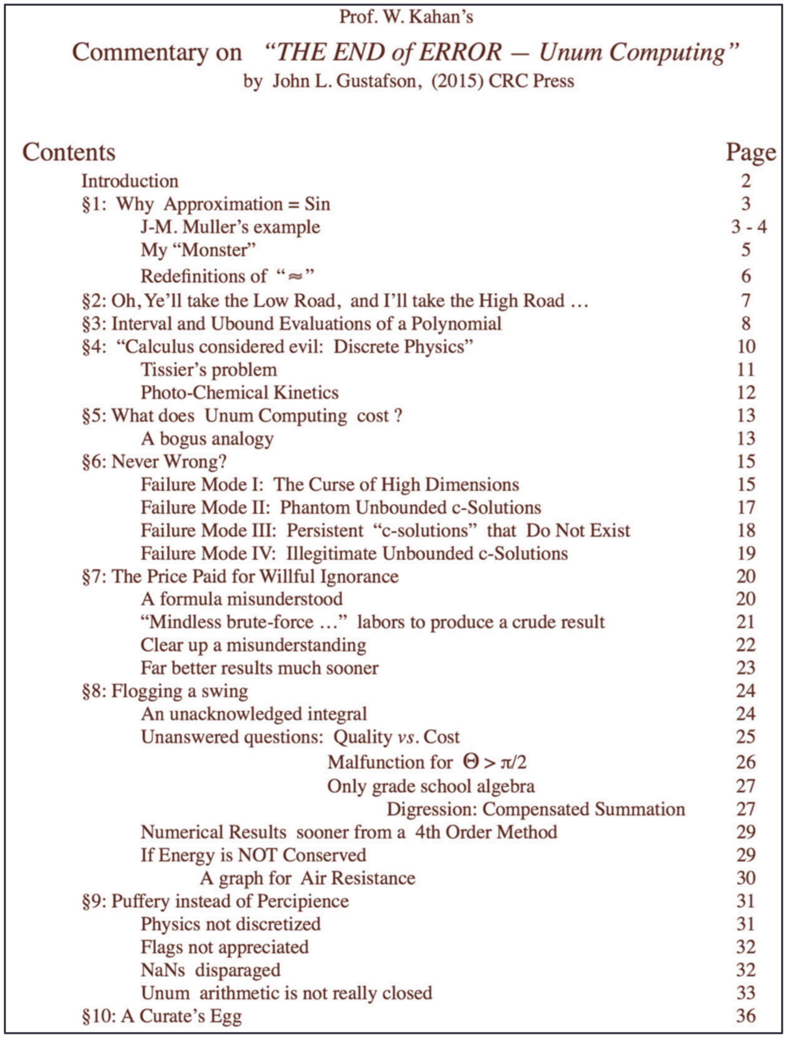

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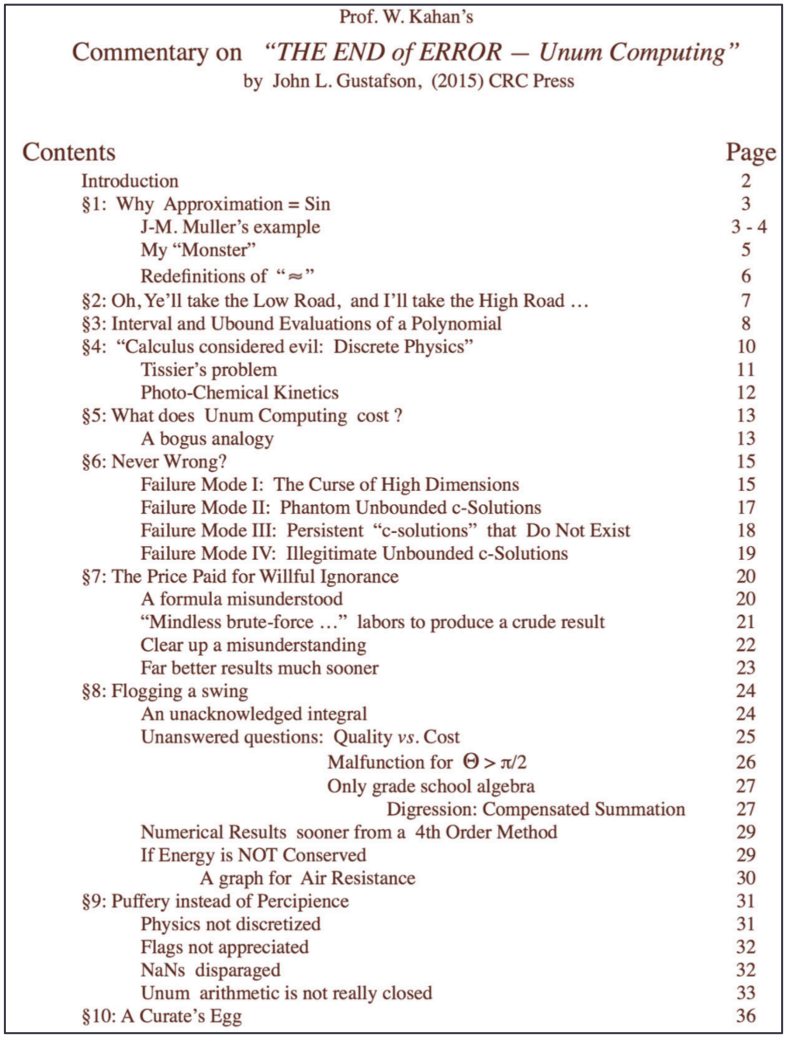

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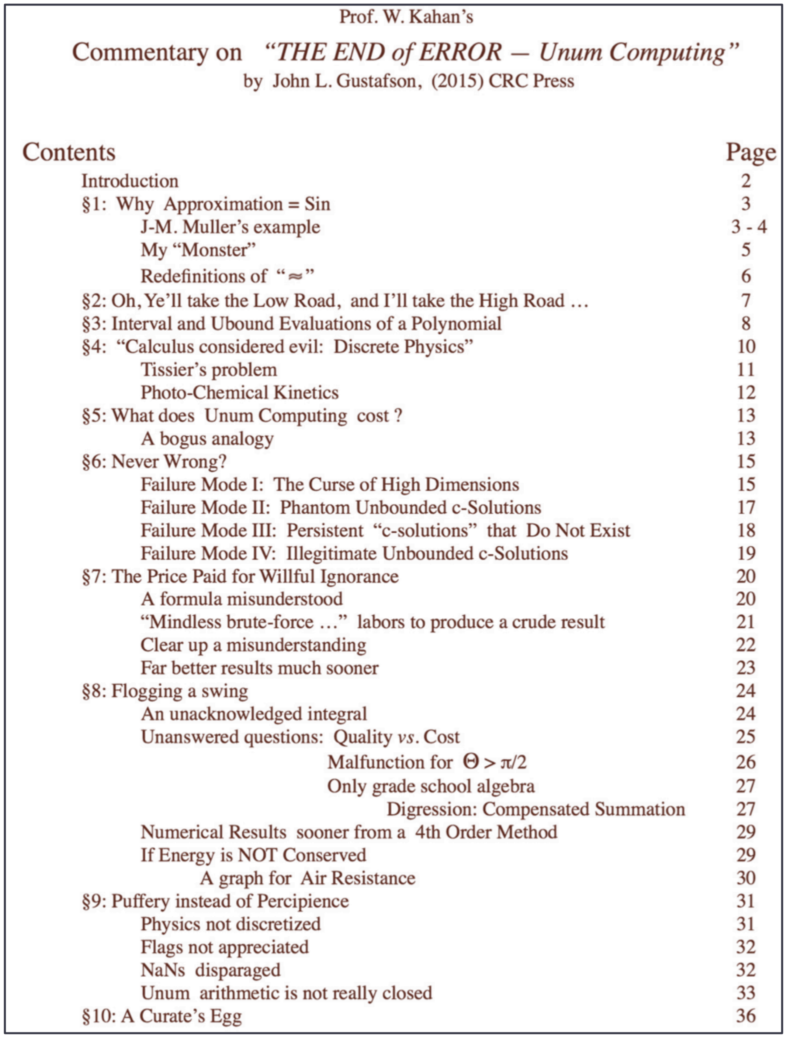

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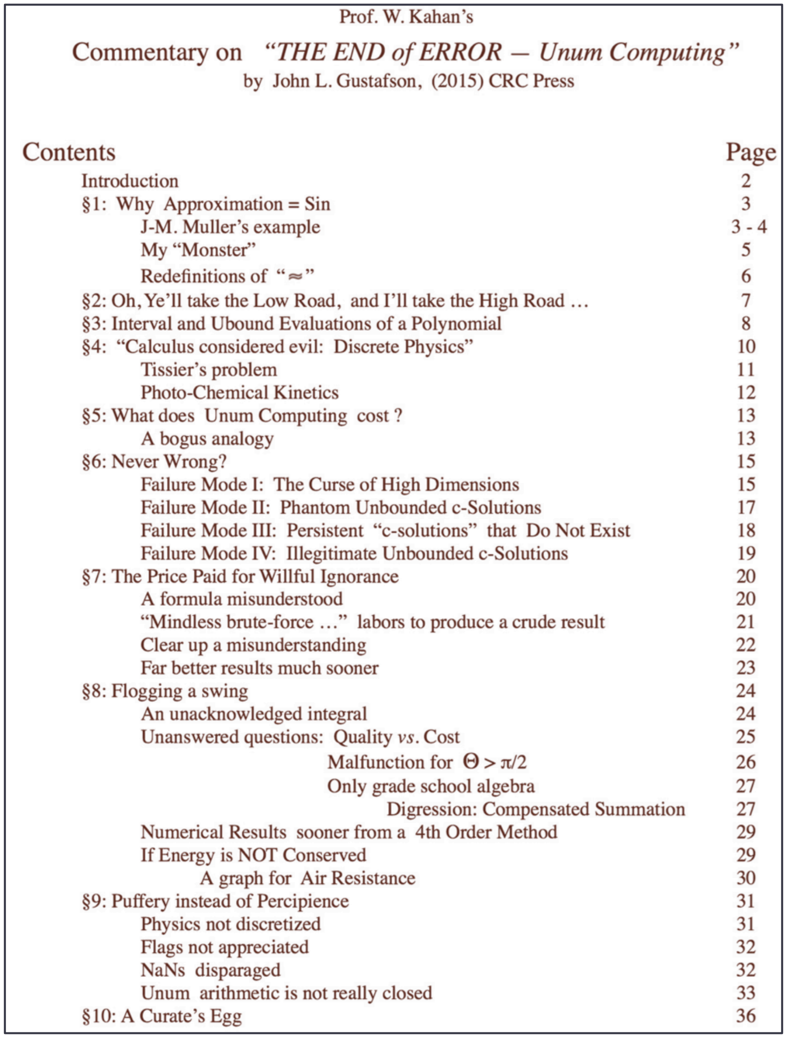

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“Bunkum! Gustafson has confused the way text is printed, or displayed on today’s bit-mapped screens, with the way text is stored in files and in DRAM memory by word-processor software. …Text stored in variable-width characters **would occupy more DRAM memory**, not less, as we shall see.” *(boldface mine)*

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Now imagine ***38 pages*** of similar attacks on things that were also not said.

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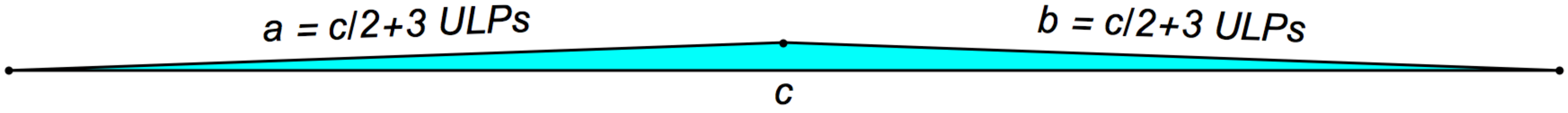
Find the area of a triangle with sides *a*, *b*, *c* where *a* and *b* are only 3 ULPs longer than half the length of *c*.



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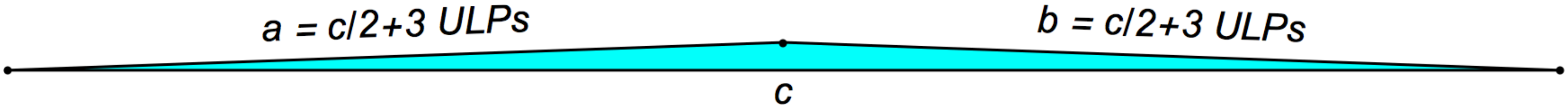

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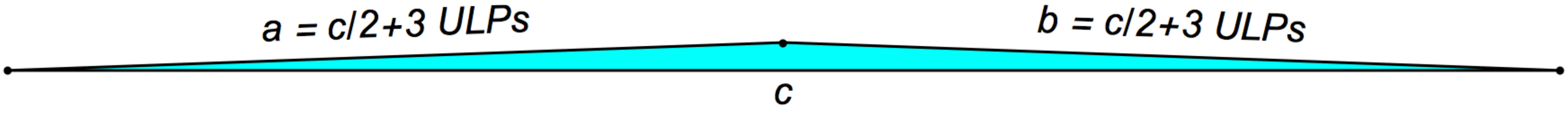

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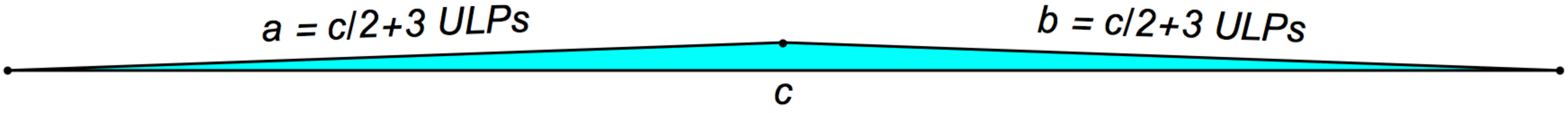

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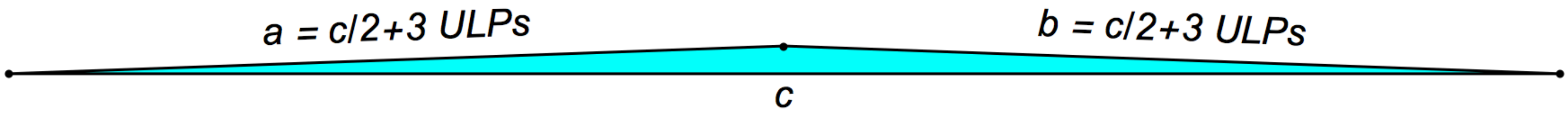

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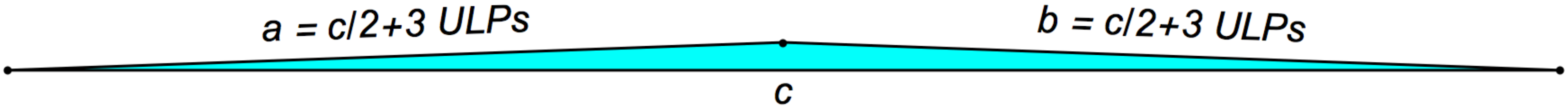

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This is within 11 ULPs of the correct area, but it takes **hours** to figure out such an approach.

It also uses twice as many operations, but that's not the issue: it's the *people cost* of the approach.

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The size of that bound is the area of a square 8 nanometers on a side.

No need to rewrite the formula.

Summary of comparison

Format Capabilities	Quad-precision IEEE floats	Unums, {4,7} environment
Dynamic Range	~6.5×10^{-4966} to 1.2×10^{4932}	~8.2×10^{-9903} to ~2.8×10^{9864}
Precision	~34.0 decimal digits	~38.8 decimal digits

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Results on thin triangle	**Quad-precision IEEE floats**	**Unums, {4,7} environment**
Maximum bits used	128	128
Average bits used	128	**90**
Result	Area **=** $3.64814908423321347259205161580577\times10^{-16}$	$3.14784204874900425235885265494550 7\times10^{-16}$ **<** Area **<** $3.14784204874900425235885265494551 4\times10^{-16}$
Type of information loss	Invisible error, very hard to debug	Rigorous bound, easy to debug if needed
Error / bound size	~4×10^{15} meters^2	~6×10^{-17} meters^2

Another “Rewrite it this way” example

From my book, to show why round-to-nearest might not be random and how unums can self-manage accuracy:

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#include < stdio.h >
float sumtester () {
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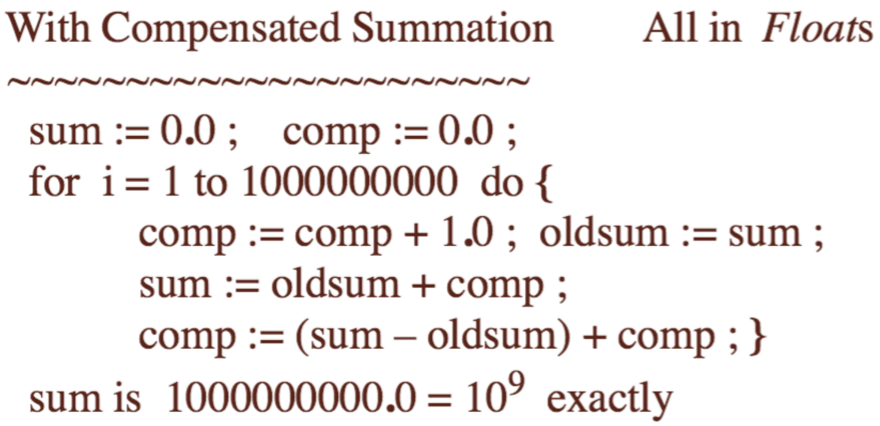
“Compensated Summation will be illustrated by application to a silly sum Gustafson uses on p. 120 to justify what unums do as intervals do, namely, convey numerical uncertainty via their widths.”

(Misreading. Actually, the example was to show how unums can automatically adjust range and precision to get the exact answer.)

Let’s try Kahan’s suggestion for $\sum_{i=1}^{n} 1$

*Screen shot from Kahan’s paper, n = **10⁹**:*



With Compensated Summation   All in *Floats*

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for  i = 1 to 1000000000  do {
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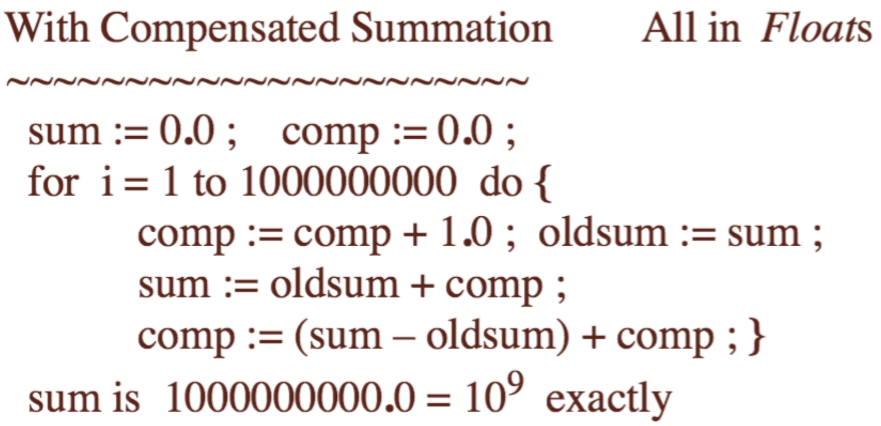

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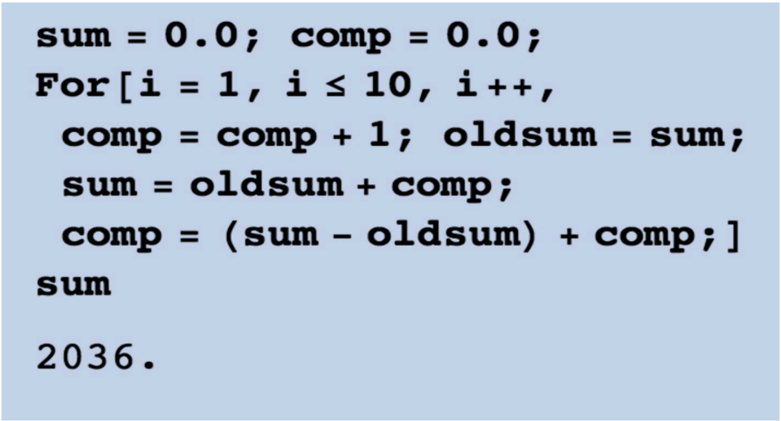

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sum = 0.0;  comp = 0.0;
For[i = 1, i ≤ 10, i++,
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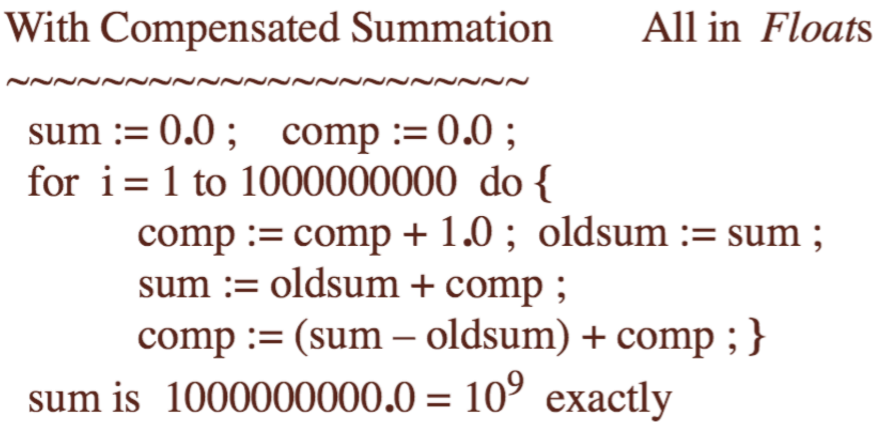

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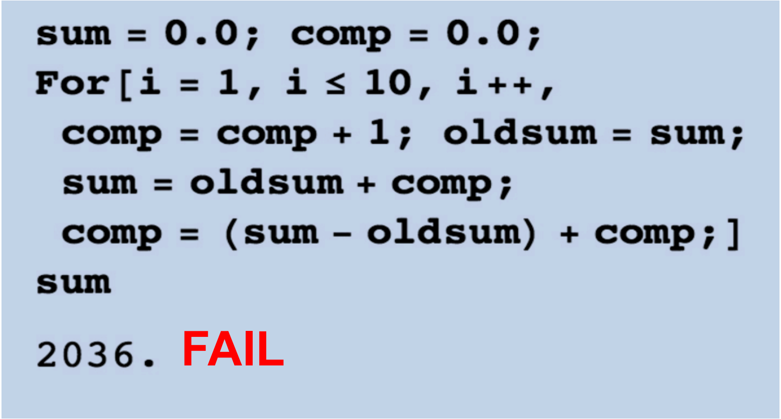

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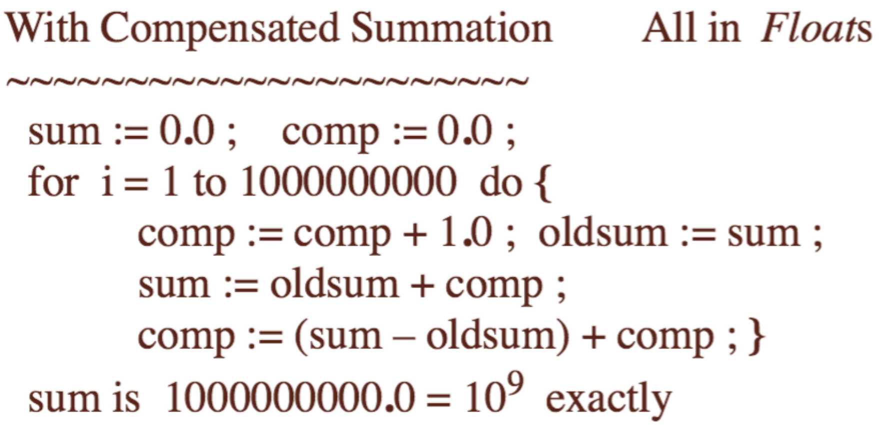

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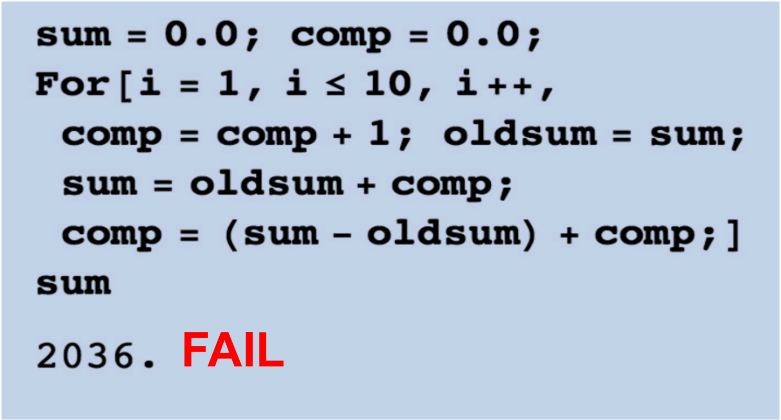

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- Rewriting code to compensate for rounding is very *error-prone*; **even Kahan didn’t get it right**.

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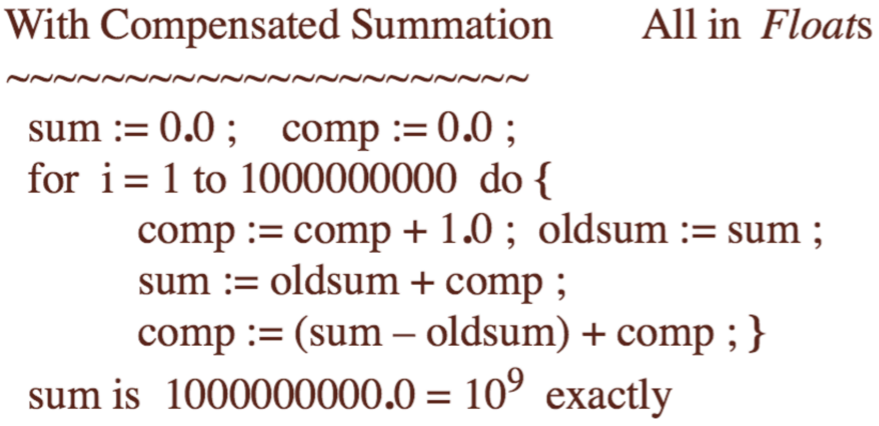
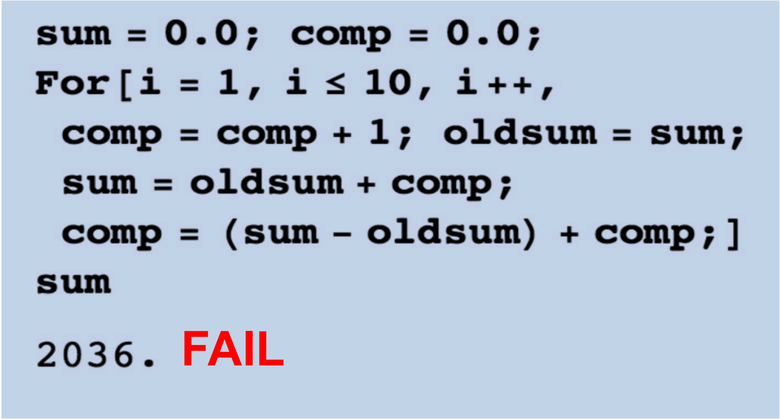


With Compensated Summation   All in *Floats*

~~~~~~~~~~~~~~~~~~~~~~~~

```
sum := 0.0 ;    comp := 0.0 ;
for  i = 1 to 1000000000  do {
         comp := comp + 1.0 ;  oldsum := sum ;
         sum := oldsum + comp ;
         comp := (sum – oldsum) + comp ; }
sum is  1000000000.0 = 10^9  exactly
```

*Screen shot from Mathematica test for sum up to n = **10***



```
sum = 0.0;  comp = 0.0;
For[i = 1, i ≤ 10, i++,
  comp = comp + 1;  oldsum = sum;
  sum = oldsum + comp;
  comp = (sum - oldsum) + comp;]
sum

2036.
```

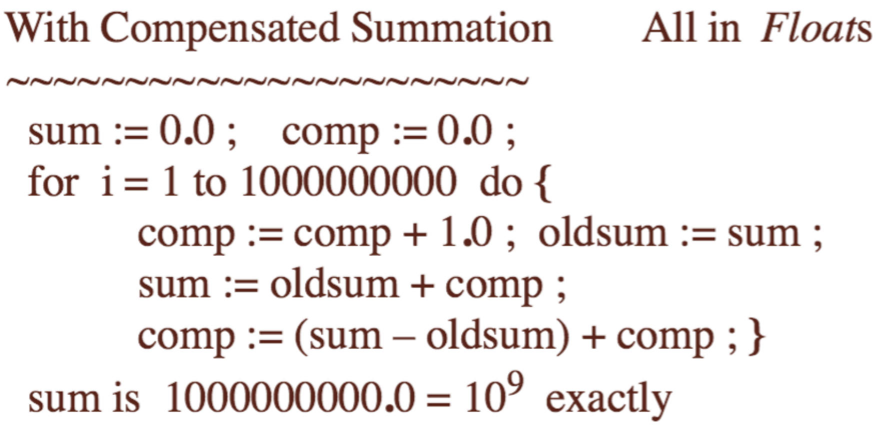
FAIL

(Attempting to sum to 10^9 gives NaN.)

- Rewriting code to compensate for rounding is very *error-prone*; **even Kahan didn’t get it right**.
- Approach uses much more human coding effort and three times as many bits to produce a wildly wrong answer.
~~~~~~~~~~~~~~~~~~~~~~~~

Let’s try Kahan’s suggestion for $\sum_{i=1}^{n} 1$

Screen shot from Kahan’s paper, n = ***10^9****:*



```
With Compensated Summation        All in  Floats
~~~~~~~~~~~~~~~~~~~~~~~~~~~~
  sum := 0.0 ;    comp := 0.0 ;
  for  i = 1 to 1000000000  do {
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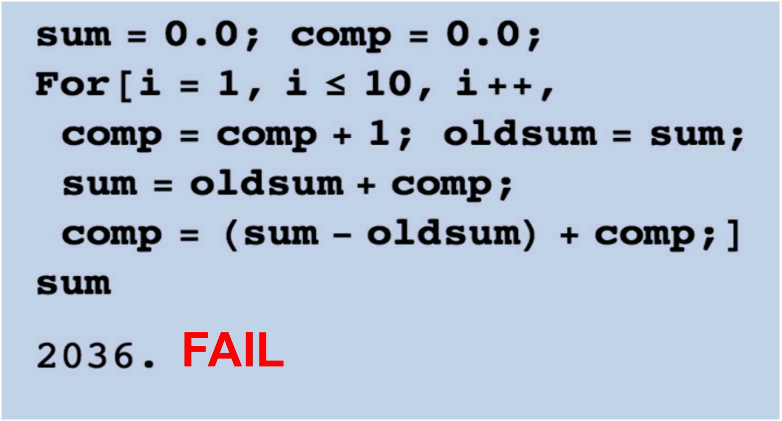

```
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For[i = 1, i ≤ 10, i++,
  comp = comp + 1;  oldsum = sum;
  sum = oldsum + comp;
  comp = (sum - oldsum) + comp;]
sum

2036.
```

FAIL

(Attempting to sum to 10^9 gives NaN.)

- Rewriting code to compensate for rounding is very *error-prone*; **even Kahan didn’t get it right**.
- Approach uses much more human coding effort and three times as many bits to produce a wildly wrong answer.
- Examples like this need to be *tested*, not merely *asserted*.

Kahan’s “Monster” Revisited

Verbatim:

```
Real variables      x, y, z ;
Real Function       T(z) := { If z = 0 then 1 else (exp(z) – 1 )/z } ;
Real Function       $Q(y) := | y - \sqrt{}(y^2 + 1) | - 1/( y + \sqrt{}(y^2 + 1) )$ ;
Real Function       $G(x) := T( Q(x)^{2} )$ ;
        For Integer n = 1 to 9999 do Display{ n , G(n) } end do.
```

“$G(x) := T(Q(x)^{2})$ ends up wrongly as 0 instead of 1 . Almost always.”

Kahan’s “Monster” Revisited

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```
Real variables      x, y, z ;
Real Function       T(z) := { If z = 0 then 1 else (exp(z) – 1 )/z } ;
Real Function       Q(y) := | y – $\sqrt{y^2 + 1}$ | – 1/( y + $\sqrt{y^2 + 1}$ ) ;
Real Function       G(x) := T( $Q(x)^2$ ) ;
     For Integer n = 1 to 9999 do Display{ n , G(n) } end do.
```

“G(x) := T($Q(x)^2$) ends up wrongly as 0 instead of 1 . Almost always.”

- Unums got exactly 1, but used “≈” (intersection test) instead of “=”.

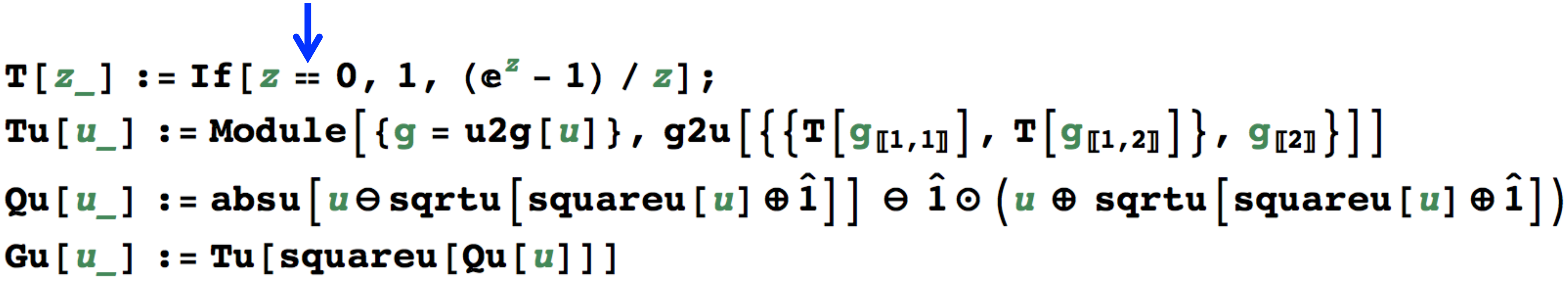
Kahan's "Monster" Revisited

Verbatim:

Real variables x, y, z ;
Real Function T(z) := { If z = 0 then 1 else (exp(z) – 1)/z } ;
Real Function Q(y) := | y – √(y^2 + 1) | – 1/(y + √(y^2 + 1)) ;
Real Function G(x) := T(Q(x)^2) ;
For Integer n = 1 to 9999 do Display{ n , G(n) } end do.

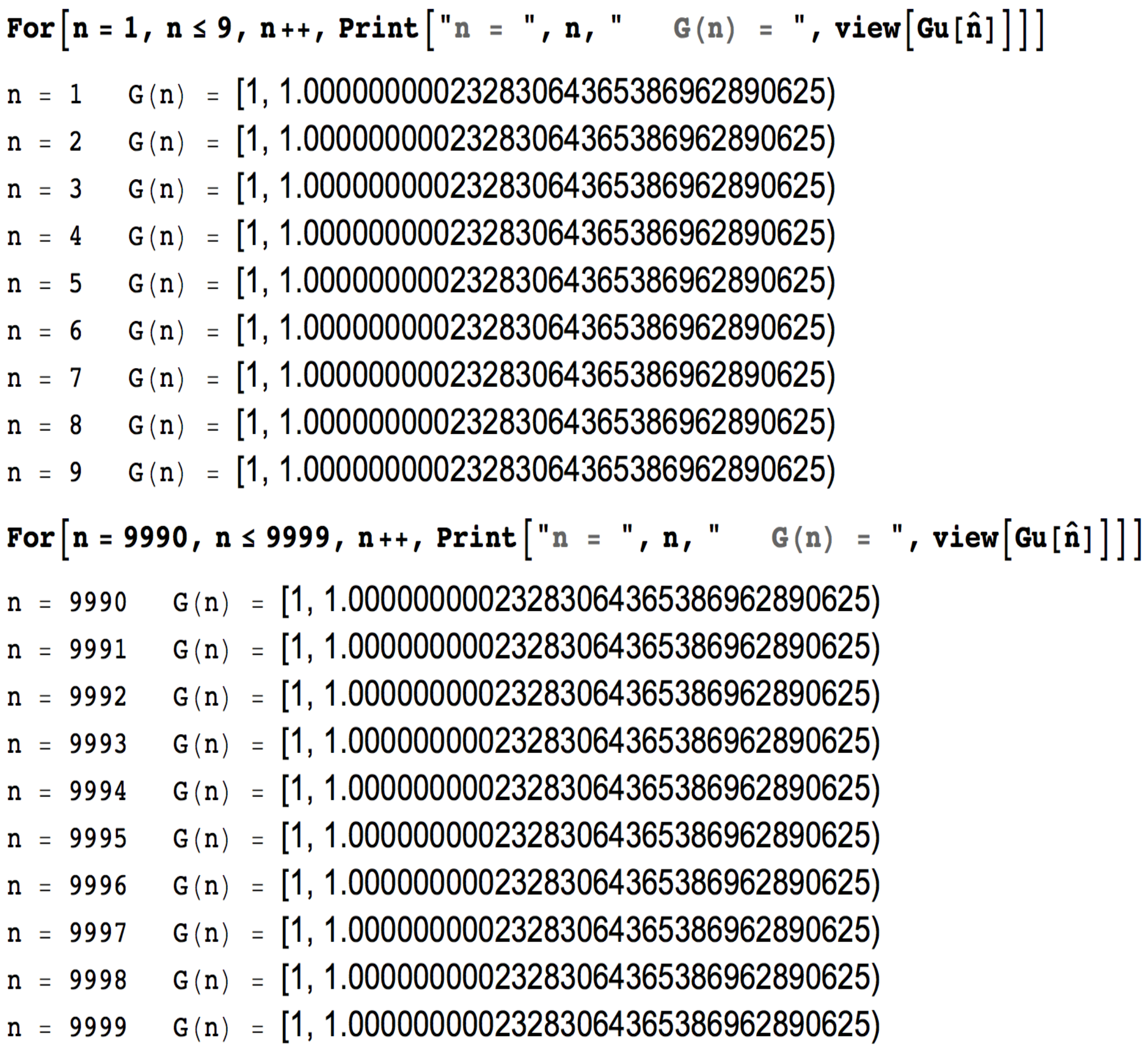
"G(x) := T(Q(x)^2) ends up wrongly as 0 instead of 1 . Almost always."

- Unums got exactly 1, but used "≈" (intersection test) instead of "=".
- Kahan cried "Foul!" so here is a unum version with exactly the specified equality test, which he says will break unums:



```
T[z_] := If[z == 0, 1, (ⅇ^z - 1) / z];
Tu[u_] := Module[{g = u2g[u]}, g2u[{{T[g[[1,1]]], T[g[[1,2]]]}, g[[2]]}]]
Qu[u_] := absu[u ⊖ sqrtu[squareu[u] ⊕ 1̂]] ⊖ 1̂ ⊙ (u ⊕ sqrtu[squareu[u] ⊕ 1̂])
Gu[u_] := Tu[squareu[Qu[u]]]
```

The result of the “=“ unum version



```
For[n = 1, n ≤ 9, n++, Print["n = ", n, "    G(n) = ", view[Gu[n̂]]]]

n = 1    G(n) = [1, 1.00000000023283064365386962890625)
n = 2    G(n) = [1, 1.00000000023283064365386962890625)
n = 3    G(n) = [1, 1.00000000023283064365386962890625)
n = 4    G(n) = [1, 1.00000000023283064365386962890625)
n = 5    G(n) = [1, 1.00000000023283064365386962890625)
n = 6    G(n) = [1, 1.00000000023283064365386962890625)
n = 7    G(n) = [1, 1.00000000023283064365386962890625)
n = 8    G(n) = [1, 1.00000000023283064365386962890625)
n = 9    G(n) = [1, 1.00000000023283064365386962890625)

For[n = 9990, n ≤ 9999, n++, Print["n = ", n, "    G(n) = ", view[Gu[n̂]]]]

n = 9990    G(n) = [1, 1.00000000023283064365386962890625)
n = 9991    G(n) = [1, 1.00000000023283064365386962890625)
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```

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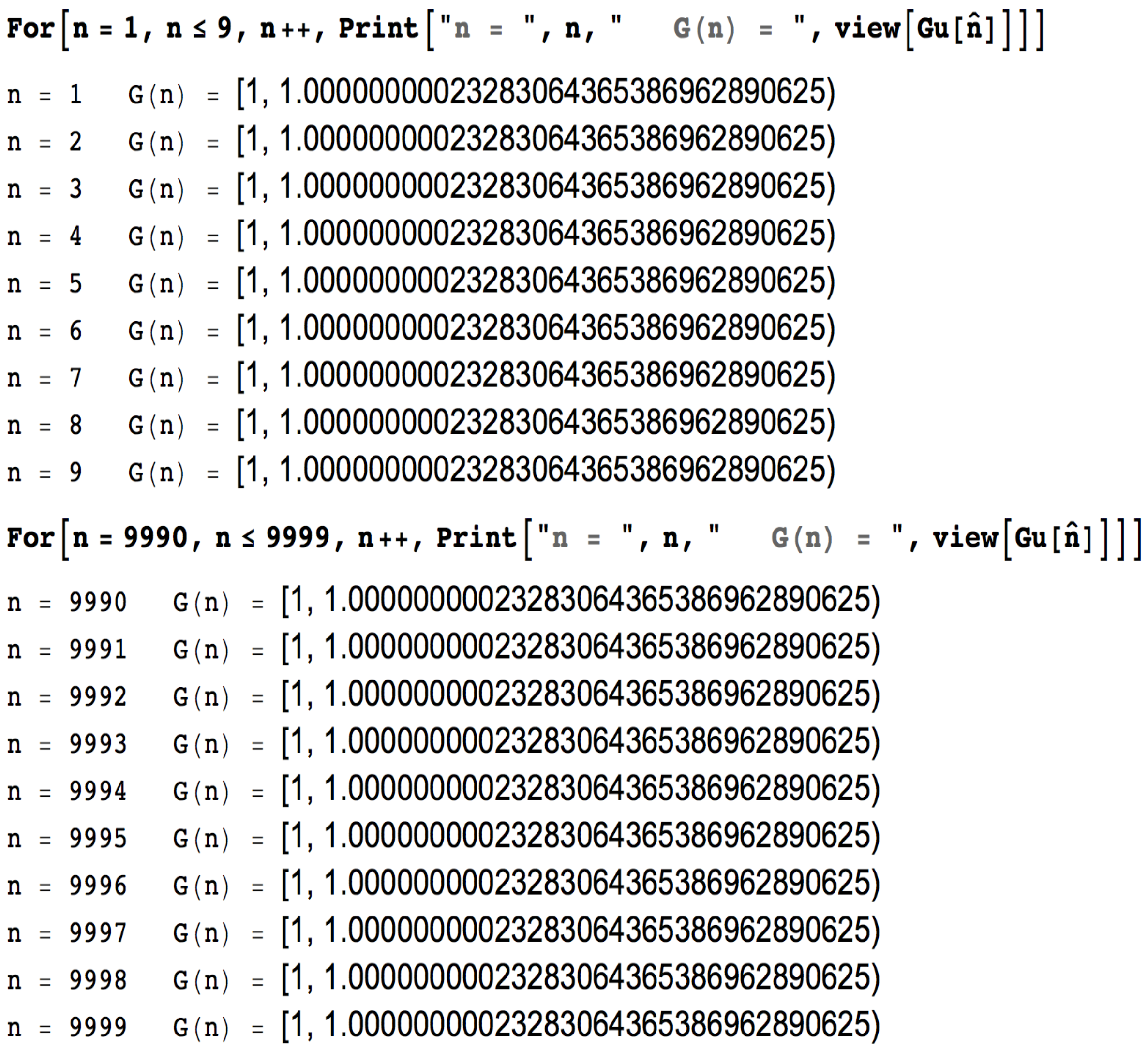

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n = 9    G(n) = [1, 1.00000000023283064365386962890625)

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n = 9999    G(n) = [1, 1.00000000023283064365386962890625)
```

Result: tight bounds, [1, 1+ε).

Never zero.

All Kahan had to do was try it. He has all my prototype code at his fingertips.

He did not *test* any of his assertions about what he thought unum arithmetic would do, but *preferred to speculate* that it would fail.

Kahan's *Unum-Targeted* Variation

```
Real Function $G^{o}(x) := T( Q(x)^2 + (10.0^{-300})^{10000\cdot(x+1)} )$ ;
For Integer n = 1 to 9999 do Display{ n , $G^{o}(n)$ } end do.
```

"Without roundoff, the ideal value $G^{o}(x) \approx 1.0$ for all real x . Rounded floating-point gets 0.0 almost always for all practicable precisions. What, if anything, does Unum Computing get for $G^{\circ}(n)$? And how long does it take? It cannot be soon nor simply 1.0 ."

Kahan’s *Unum-Targeted* Variation

Real Function Gº *(*x*)* := T*(* Q*(*x*)*^{2} + (10.0^{-300})$^{10000\cdot(x+1)}$ *)* ;
For Integer n = 1 to 9999 do Display{ n , Gº *(*n*)* } end do.

“Without roundoff, the ideal value Gº (x) ≈ 1.0 for all real x . Rounded floating-point gets 0.0 almost always for all practicable precisions. What, if anything, does Unum Computing get for G°(n*)* ? And how long does it take? It cannot be soon nor simply 1.0 .”

Surprise. Unums handled this without a hiccup. Quickly.

Kahan’s *Unum-Targeted* Variation

Real Function G^{o} *(x)* := T(Q*(*x*)*^{2} + $(10.0^{-300})^{10000\cdot(x+1)}$ *)* ;
For Integer n = 1 to 9999 do Display{ n , G^{o} *(*n*)* } end do.

“Without roundoff, the ideal value G^{o} (x) ≈ 1.0 for all real x . Rounded floating-point gets 0.0 almost always for all practicable precisions. What, if anything, does Unum Computing get for G^{o}*(*n*)* ? And how long does it take? It cannot be soon nor simply 1.0 .”

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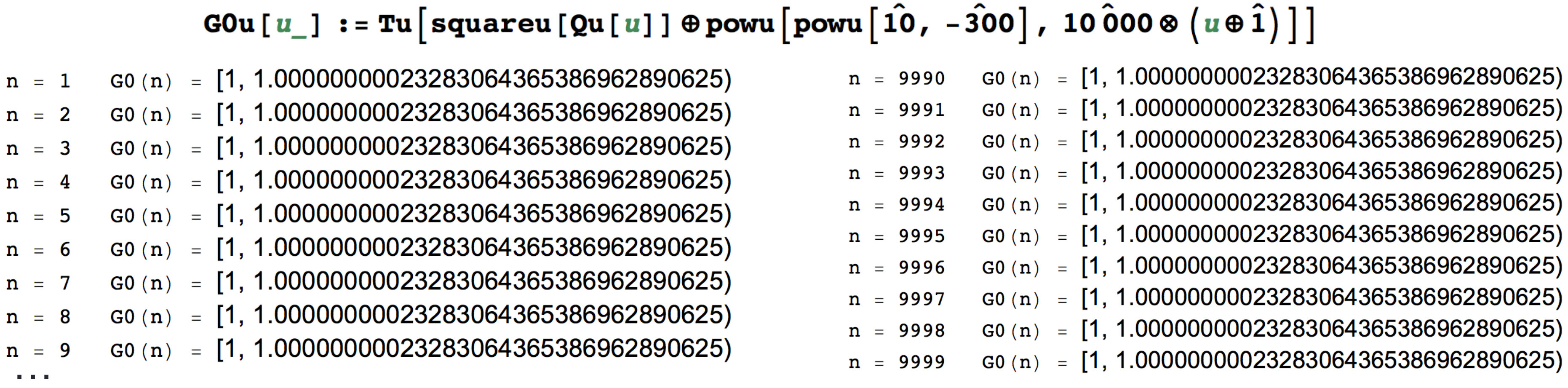

```
G0u[u_] := Tu[squareu[Qu[u]] ⊕ powu[powu[1̂0, -3̂00], 10 0̂00 ⊗ (u ⊕ 1̂)]]
```

```
n = 1    G0(n) = [1, 1.00000000023283064365386962890625)
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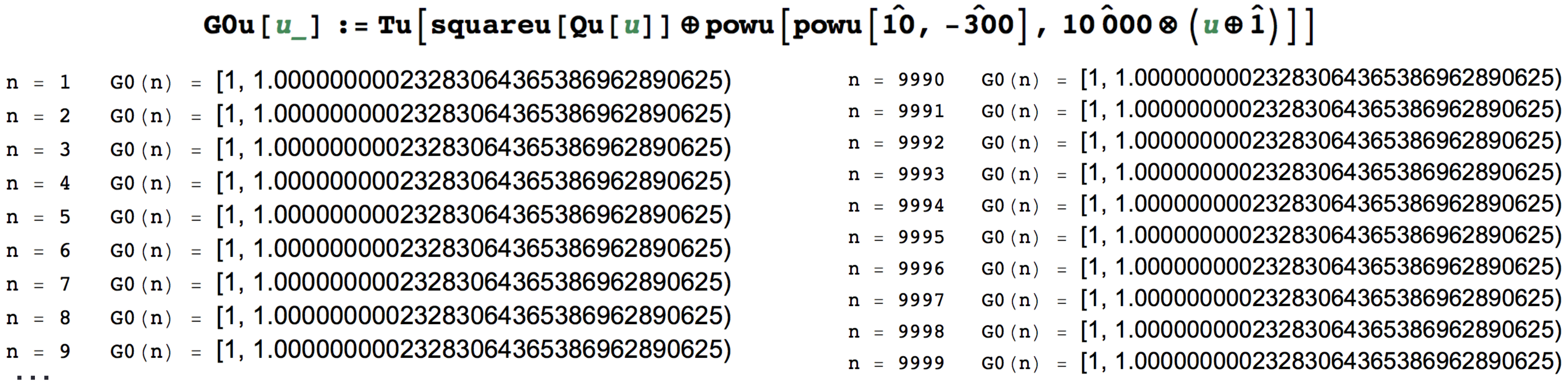

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 ...                                                          n = 9999   G0(n) = [1, 1.00000000023283064365386962890625)
```
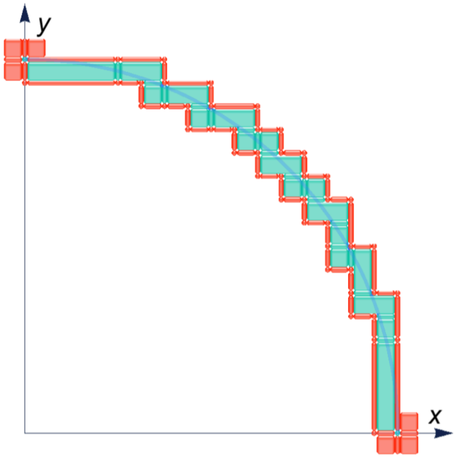

Kahan’s “infinitesimal” (his term) becomes unum (0, ε).

An Inconvenient Infinity

My example of quarter-circle integration takes $O(n)$ time for n subdivisions, and produces $O(1/n)$ size rigorous bounds. Works on any continuous function.



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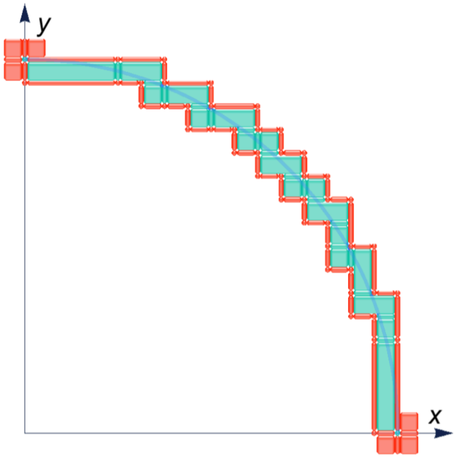

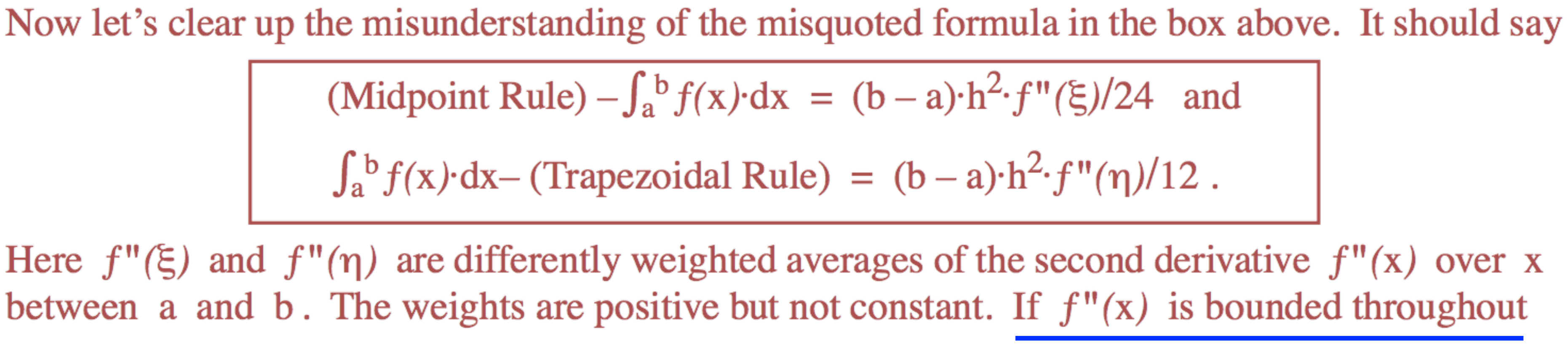

Now let’s clear up the misunderstanding of the misquoted formula in the box above. It should say

$$\text{(Midpoint Rule)} - \int_a^b f(x)\cdot dx \;=\; (b-a)\cdot h^2\cdot f''(\xi)/24 \quad \text{and}$$

$$\int_a^b f(x)\cdot dx - \text{(Trapezoidal Rule)} \;=\; (b-a)\cdot h^2\cdot f''(\eta)/12\ .$$

Here $f''(\xi)$ and $f''(\eta)$ are differently weighted averages of the second derivative $f''(x)$ over x between a and b. The weights are positive but not constant. If $f''(x)$ is bounded throughout

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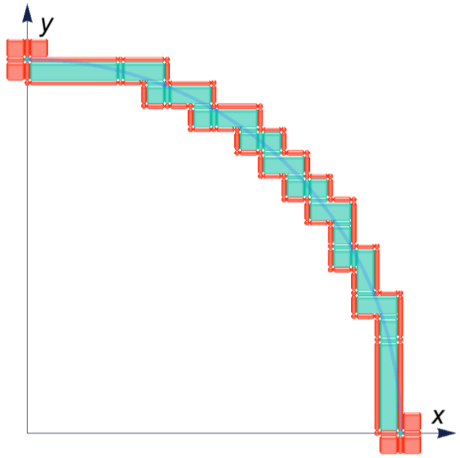

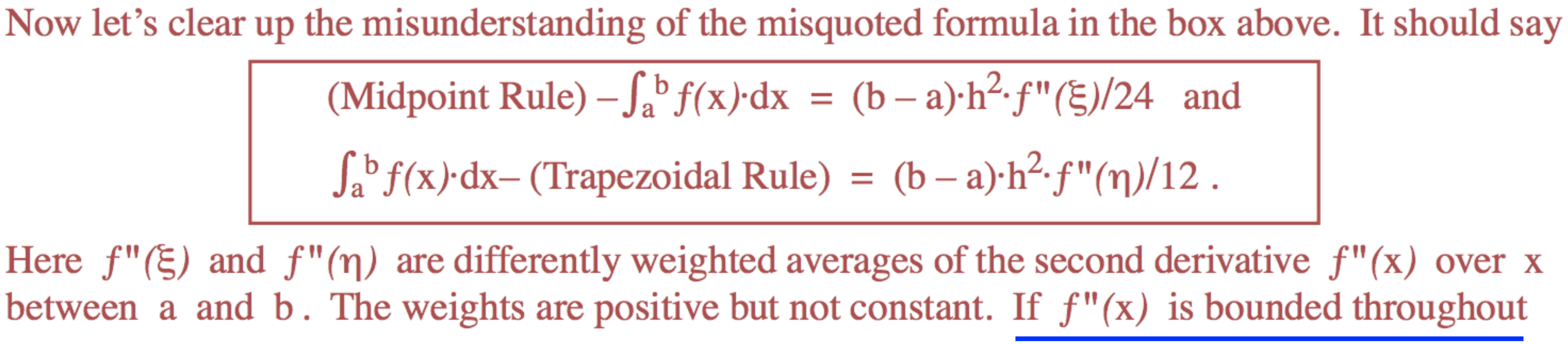

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Here $f''(\xi)$ and $f''(\eta)$ are differently weighted averages of the second derivative $f''(x)$ over x between a and b. The weights are positive but not constant. If $f''(x)$ is bounded throughout

But $f''(x)$ is *not* bounded throughout. ***Kahan uses the formula anyway!***

Also, Kahan says my method is $O(n^2)$.
Willful misunderstanding. Obviously not true (see figure above).

Too many mistakes to cover here…

The book claims it ends all error.

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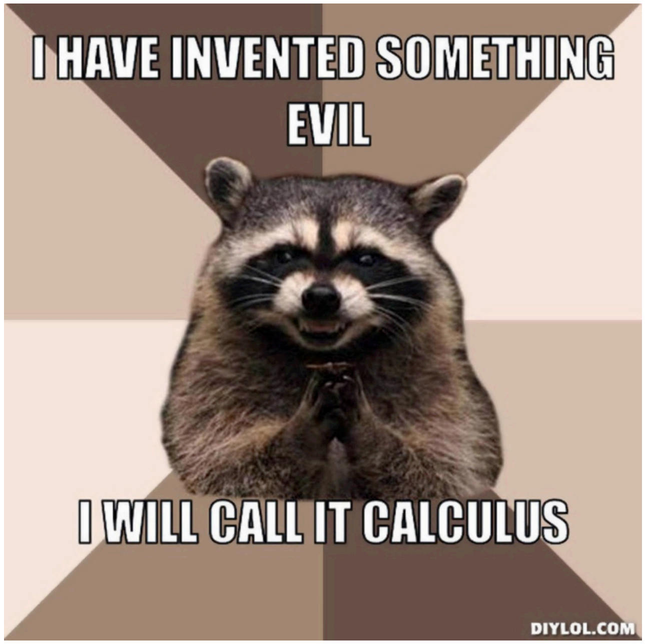

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Unums will cost *thousands* of extra transistors!	Which will cost *thousandths* of a penny. The year is 2016, not 1985.
His approach is very inefficient; here's a faster one that usually works.	

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Unums will cost *thousands* of extra transistors!	Which will cost *thousandths* of a penny. The year is 2016, not 1985.
His approach is very inefficient; here's a faster one that usually works.	I'm not interested in methods that *usually* work. We have plenty of those.

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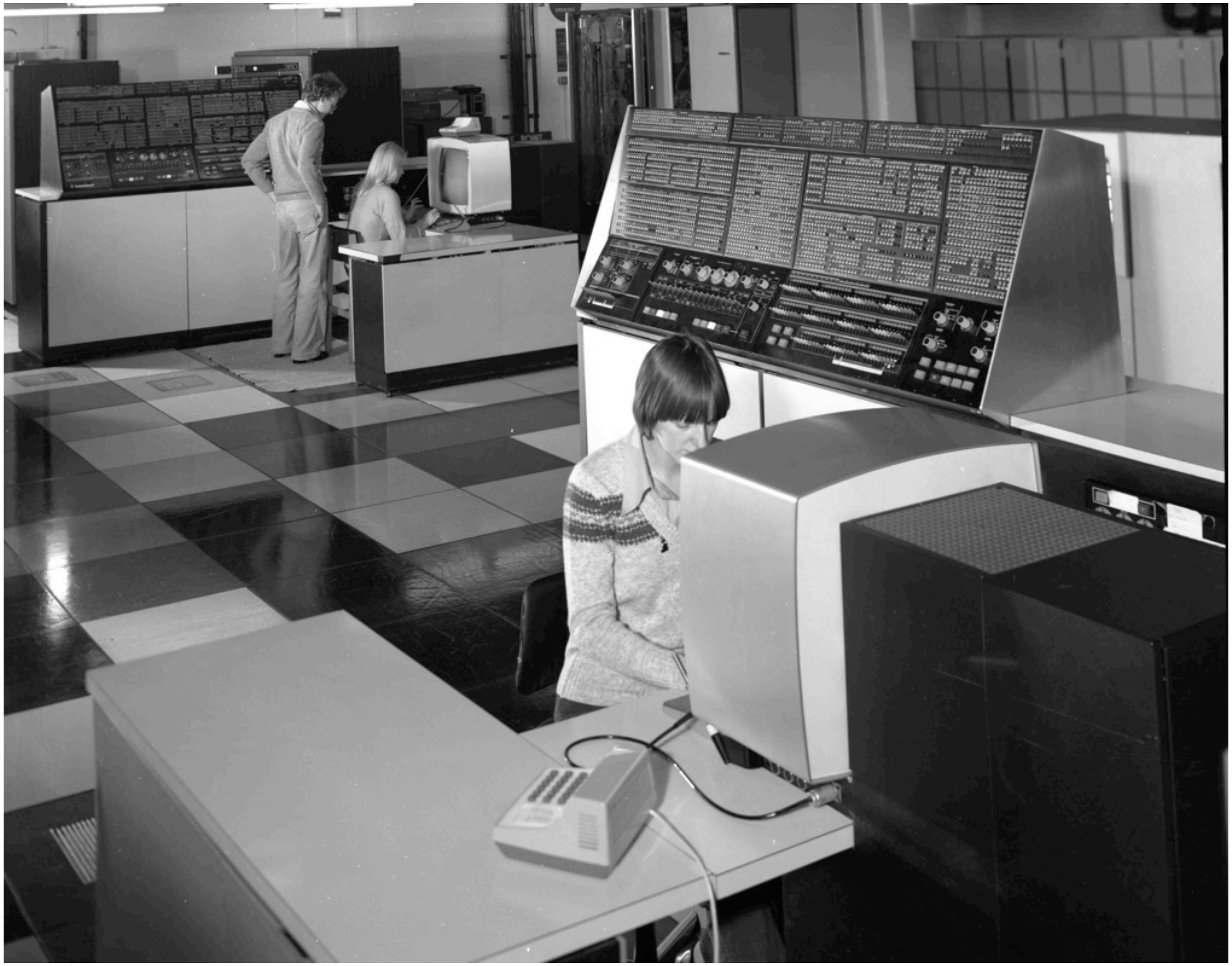
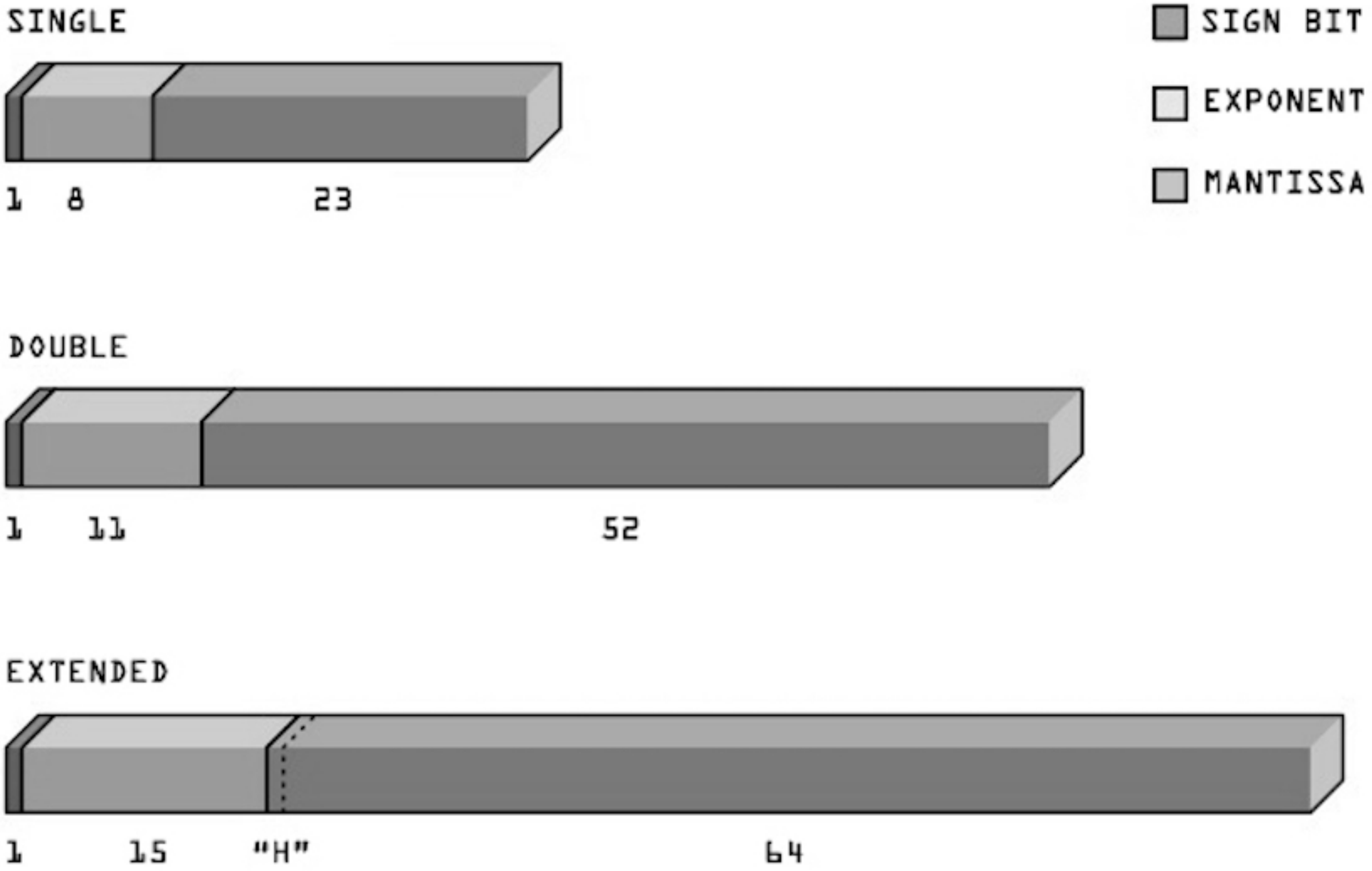
COMPUTERS THEN



COMPUTERS NOW

ARITHMETIC THEN



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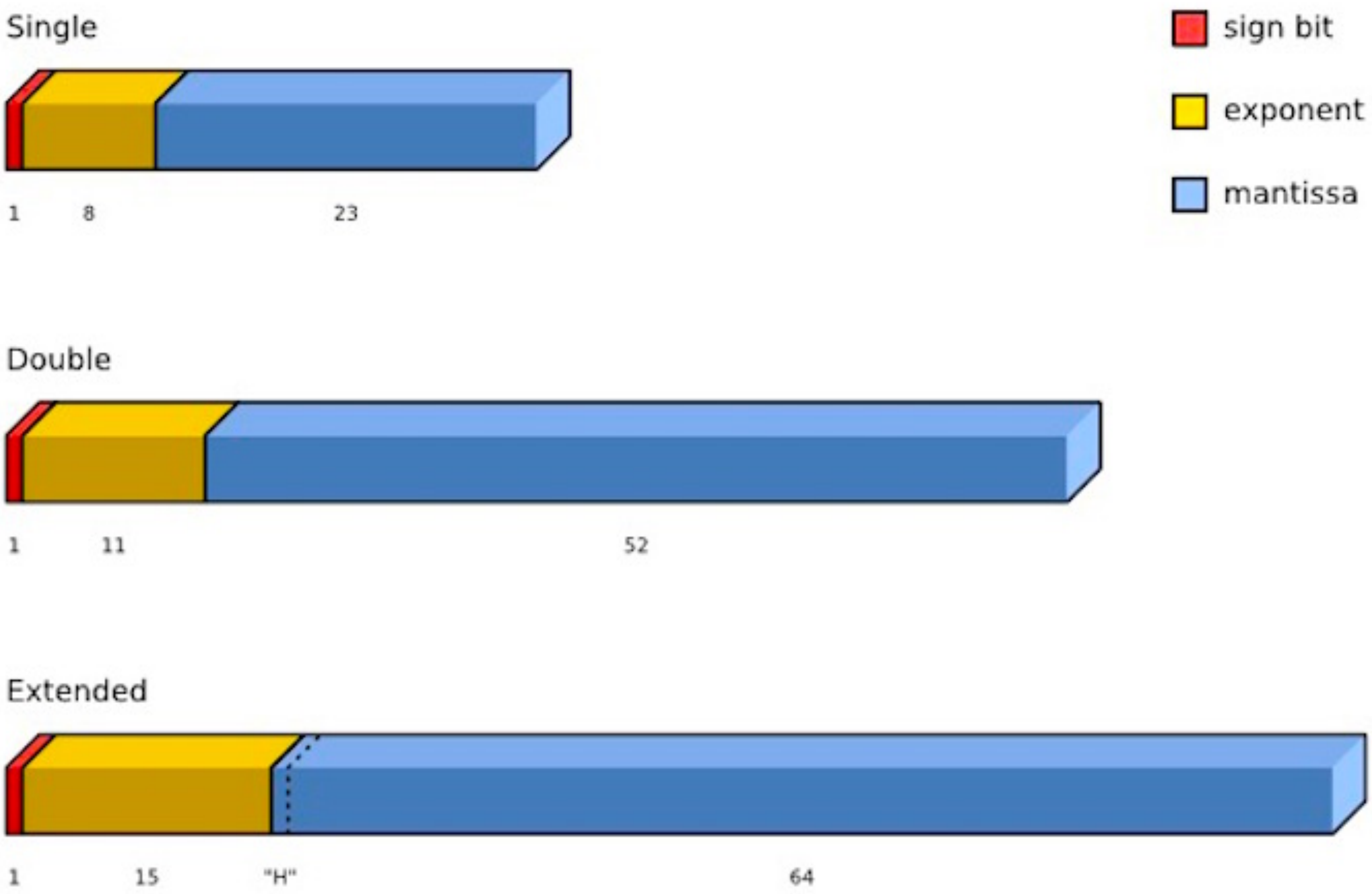

Kahan’s biggest blind spot of all

Remember: There is nothing floats can do that unums cannot. ■

The last line of my book, p. 413, and emphasized throughout

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- Float methods are a good way to deal with “The Curse of High Dimensions” in many cases, like getting a starting answer for *Ax* = *b* linear systems in polynomial time.

WK's Dysphemisms, Insults, and Rants about *The End of Error: Unum Computing*

Bunkum!

Lies

liar

Bogus

perverse

crude

WK’s Dysphemisms, Insults, and Rants about *The End of Error: Unum Computing*

foolish

Bunkum!

Lies

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Invective worked for Donald Trump, but… is this really the right way to discuss *mathematics*?

“THE LORD OF THE REALS…
DOES NOT SHARE POWER.”