Epistemological Obstacles in Coming to Understand the Limit Concept at Undergraduate Level: A Case of the National University of Lesotho

Eunice Kolitsoe Moru

A thesis submitted in fulfillment of the requirements for the degree of Doctor of Philosophy in the School of Science and Mathematics Education in the Faculty of Education, University of the Western Cape.

Supervisors: Professor Jan Persens, University of the Western Cape, South Africa
Professor Trygve Breiteig, Agder University College, Norway

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Keywords

Epistemological obstacles

Understanding

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Limit concept

Sequence

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Function

APOS theory

Semiotics

Abstract

The purpose of this study was to investigate the epistemological obstacles that mathematics students at undergraduate level encounter in coming to understand the limit concept. The role played by language and symbolism in understanding the limit concept was also investigated. A group of mathematics students at undergraduate level at the National

University of Lesotho (NUL) was used as the sample for the study. Empirical data were collected by using interviews and questionnaires. These data were analysed using both the APOS framework and a semiotic perspective.

Within the APOS framework, the pieces of knowledge that have to be constructed in coming to understand the limit concept are actions, processes and objects. Actions are interiorised into processes and processes are encapsulated into objects. The conceptual structure is called a schema. In investigating the idea of limit within the context of a function some main epistemological obstacles that were encountered when actions were interiorised into processes are over-generalising and taking the limit value as the function value. For example, in finding the limit value L for $f(x)$ as x tends to 0, 46 subjects out of 251 subjects said that they would calculate $f(0)$ as the limit value. This method is appropriate for calculating the limit values for continuous functions. However, in this case, the method is generalised to all the functions. When these subjects encounter situations in which the functional value is equal to the limit value, they take the two to be the same. However, the two are different entities conceptually.

Within the context of a sequence everyday language acted as an epistemological obstacle in interiorising actions into processes. For example, in finding $\lim_{x\to\infty}\frac{(-1)^n}{n}$, the majority of the subjects obtained the correct answer 0. It was however revealed that such an answer was obtained by using an inappropriate method. The subjects substituted one big value for n in the formula. The result obtained was the number close to 0. Then 0 was taken as the limit value because the subjects interpreted the word 'approaches' as meaning 'nearer to'. Other subjects rounded off the result. In everyday life when one object approaches another, we might say that they are nearer to each other. It seems that in this case the subjects used this meaning to get 0 as the limit value. We also round off numbers to the nearest unit, tenth, etc. The limit value is however a unique value that is found by using the limiting process of 'tending to' or 'approaching' which requires infinite values to be considered. Some are computed and others are contemplated.

In constructing the coordinated process schema, $f(x) \to L$ as $x \to a$, over-generalisation and everyday language were still epistemological obstacles. Subjects still perceived the limit value to exist where the function is defined. The limit was also taken as a bound, lower or upper bound. In a case where the function was represented in a tabular form, the first and the seemingly last functional value that appeared in the table of values were chosen as the limit values. Limit values were also approximated. In constructing the coordinated process $a_n \to L$ as $n \to \infty$, representation, generalisation and everyday language also acted as epistemological obstacles. An alternating sequence was perceived as not one but two sequences. Since the subjects will have met situations where convergence means meeting at a point, as in the case of rays of light, a sequence was said to converge to a number that did not change in the given decimal digits. For example, the limit of the sequence {3.1, 3.14, 3.141, 3.1415, …} was taken to be 3 or 3.1 as these are the digits that are the same in all the terms.

In encapsulating processes into objects, everyday language also acted as an epistemological obstacle. When subjects were asked what they understood the limit to be, they said that the limit is a boundary, an endpoint, an interval, or a restriction. Though these interpretations are correct they are however, inappropriate if used in the technical context such as the mathematical context. While some subjects referred to the limit as a noun to show that they refer to it as an object, other subjects described the limit in terms of the processes that give rise to it. That is, it was described in terms of either the domain process or the range process. This is an indication that full encapsulation of processes into objects was not achieved by the subjects.

The role of language and symbolism has been identified in making different connections in building the concept of limit as: representation of mathematical objects, translation between modes of representation, communication of mathematical ideas, manipulation of surface or syntactic structures and the overcoming of epistemological obstacles. In representation some subjects were aware of what idea some symbolism signified while other subjects were not. For example, in the context of limit of a sequence, most subjects took the symbolism that represented an alternating sequence, $a_n = (-1)^n$, to represent two sequences. The first sequence was seen as {1, 1, 1, 1,…} and the second as {-1, -1, -1, -1,…}.This occurred in all modes of representation.

In translating from one mode of representation to another, the obscurity of the symbol $\lim_{x \to a} f(x) = L$ was problematic to the students. This symbol could not be related to its equivalent form $f(x) \to L$ as $x \to a$. The equal sign, '=', joining the part $f(x)$ and L does not reflect the process of $f(x)$ tending to L, rather it appears as if it is the functional value that is equal to L. Hence, instead of looking for the value that is approached the subjects chose one of the given functional values. The part of the symbol $\lim\limits_{x \to a}$ was a source of difficulty in translating the algebraic form to the verbal or descriptive. The subjects saw this part to mean "the limit of x tends to a" rather than seeing the whole symbolism as the limit of $f(x)$ as x *t*ends to a. Some subjects actually wrote some formulae in the place of L because of this structure, e.g., $\lim\limits_{x \to a} f(x) = 2x$. These subjects seemed to have concentrated on the part $f(x) = ...$ This is probably because they are used to situations where this symbolism is used in representing functions algebraically.

In communicating mathematical ideas the same word carried different meanings for the researcher and for the subjects in some cases. For example, when the subjects were asked what it means to say a sequence diverges, one of the interpretations given was that divergence means tending to infinity. So, over-generalisation here acted as an epistemological obstacle. Though a sequence that tends to infinity diverges, this is not the only case of divergence that exists and therefore cannot be generalised in that way.
The manipulation of the surface structures was done instrumentally by some subjects. For example, in finding $\lim\limits_{x \to 0} \frac{\sqrt{x^2+9}-3}{x^2}$, during the manipulation some subjects obtained part of the expressions such as $\frac{x}{2x}$ by rationalising or $\frac{x^2}{x^2}$ by using L'Hospital's rule which needed to be simplified. Instead of simplifying the expressions further at this stage, the substitution of 0 was done. So, $\frac{0}{0} = 0$ was obtained as the answer. This shows that neither the reasons for performing the manipulations, nor the process of rationalising for example was understood. The result was still an indeterminate form of limit. The numerator was also not yet in a rational form.

In using language to overcome epistemological obstacles, subjects were exposed to a piece of knowledge that falsified the knowledge they had so that they could rethink replacing the old with the new. In some cases, this was successful but in others, the subjects did not surrender these old pieces of knowledge. For example, when asked what they understood the ‘rate of change’ to mean, the majority of the subjects associated the rate of change with time only. However, when referred to a situation that required them to find the rate of change of an area with respect to radius, some subjects changed their minds but others did not. Those who did not change their minds probably did not make any connections between ideas under discussion.

The implications for practice of the findings include: In teaching one should discuss explicitly how answers to tasks concerning limits are obtained. The idea of the limit value as a unique value can only be recognized if the process by which it is obtained is discussed. It should not be taken for granted that students who respond correctly understand the answers. It is evident from the study that even when correct answers are given, improper methods may have been used. Hence, in investigating epistemological obstacles attention should also be paid to correct answers. Also beyond this, students should be exposed to different kinds of representation of the limit concept using simple functions and using a variety of examples of sequences. Words with dual or multiple meaning should also be discussed in mathematics classrooms so that students may be aware of the meanings they carry in the mathematical context. Different forms of indeterminate states of limit should be given attention. Relations should also be made between the surface structures and the deeper structures.

Declaration

I declare that *Epistemological Obstacles in Coming to Understand the Limit Concept at Undergraduate Level: A Case of the National University of Lesotho* is my own work, that it has not been submitted before for any degree or examination in any other university, and that all the sources I have used or quoted have been acknowledged as complete references.

Eunice Kolitsoe Moru

September 2006

Signed ………………………………..

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To my students at NUL I love you and thank you for having given me the opportunity to pursue a study, which I know most of you cherished very much. Our interaction was full of laughter and great appreciation for what we were all reaping from it. You are a very special group of people, not only because you opened your hearts for me to investigate some obstacles to your success in understanding a unique concept, the limit concept, in the study of calculus, but also because you are the best thing that ever happened to me in my career. I always feel good whenever I discover that our interaction has indeed changed you for the better. No words can explain the role you played in this journey.

Dedication

I would also like to dedicate this thesis to the following:

To my mother whom I know that the pain of my father's loss always pricks like a thorn. Despite this, she has been there for me.

To my brothers and sisters, we have a very special bond, which no one but us only can understand. Your company, your warmth and passion have carried me across.

To my two sons, Toka and Molise, you are a blessing. I will forever be grateful for having you in my life. Toka, my son and my friend, when days were tough you were always by my side. You gave me strength to persevere.

To the memory of my father, Edwin Dutton Moru, no one can fill up the empty space you have left in my heart and soul. This piece of work has been produced to ease the unbearable pain of losing you. I love you daddy.

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Introduction

This chapter begins by giving some background information on the topic of the study. This is followed by reasons that led to pursuing the study. The last part of the chapter presents the questions that will be addressed by the study.

Background to the study

Problems of understanding fundamental calculus concepts by students in tertiary education colleges and universities are evidenced by a body of research studies conducted in different parts of the world, e.g., the United Kingdom (Tall & Schwarzenberger, 1978), the United States of America (Davis & Vinner, 1986; Aspinwall & Miller, 2001), Poland (Sierpinska, 1987), Sweden (Juter, 2003a, 2003b, 2004, 2005), and South Africa (Bezuidenhout, 2001; Kannemeyer, 2003). The researchers have identified, classified and analysed these problems from historical, epistemological, and learning theory perspectives. History is important because mathematical concepts are a result of the developments of the past. The way knowledge is acquired is an epistemological issue and the major purpose of learning is to acquire knowledge. Hence, these three perspectives qualify to be used as lenses in understanding problems that students encounter in a learning situation.

Closely related to this world wide research concern, are attempts that have been made to alleviate the problems in education and schooling. Some examples of such efforts are:

- Introduction of the 'Calculus Reform Movement' which started in the United States of America and later spread to other parts of the world;
- Emphasis on employing alternative methods complementing traditional lectures in teaching calculus;
- Introduction of calculus to students before tertiary level;
- Development of calculus textbooks;

- Introduction of programs connected to the use of computer-technology in calculus classrooms; and
- The introduction of portable graphic calculators with graphic, numeric, and symbolic facilities (Tall, 1996).

In spite of these educational efforts, the study of calculus still causes problems in mathematics education today (Taback, 1975, Tall & Schwarzenberger, 1978; Tall & Vinner, 1981; Orton, 1983a, 1983b; Davis & Vinner, 1986; Sierspinska, 1987; Cornu, 1991; Monaghan, 1991; Williams, 1991; Cottrill *et al*., 1996; Tall, 1996; White & Mitchelmore, 1996; Billings & Klanderman; 2000; Aspinwall & Miller, 2001; Bezuidenhout, 2001; Kannemeyer, 2003; Juter, 2003a, 2003b, 2004, 2005). Lesotho is no exception to these problems.

Lesotho has only one university, the National University of Lesotho (NUL). Students join the university after the completion of a 7-3-2 (7 years of primary education, 3 years of secondary school education, and 2 years of high school education) system of education. As in other parts of the world, for example, United States of America, Canada, South Africa, Kenya, and United Kingdom, most students enter the university at the age of eighteen. While a good grade in English Language is required to qualify for entry into other faculties, the Faculty of Science and Technology (FOST) admits students who have obtained a minimum pass grade in English Language. English is the medium of instruction in higher levels of education in Lesotho. Students' performance can therefore be affected by it.

Students admitted in FOST undergo a Pre-Entry Science Programme (PESP), which lasts for about 10 weeks (May to July). The major purpose of PESP is to bridge the gap between the high school and the university content (see Appendix A for the course outline). The undergraduate science degree programme takes four years. An academic year consists of two semesters. The first semester starts in August and ends in December. The second semester starts in January and ends in July the following calendar year. During the first semester, the lectures end in November and the students sit for the end of semester examinations in December. During the second semester, the lectures end in April and the students sit for the examination in May. The months of June and July are devoted to the marking and the processing of the end of year results.

In the first year of study, students in FOST register for two mathematics courses, a pre-calculus and a calculus course. During the first semester they do a pre-calculus course (Algebra, Trigonometry and Analytic Geometry), which covers logic and proof, finite sequences and series (see Appendix B for course outline). The pre-calculus content background knowledge of students from Lesotho high schools include sequences whose terms are connected to a rule or formula, finding the inverse of a given function, finding the gradient of a straight line graph, calculating distance as area under a linear speed-time graph, and estimating the gradient of the curve at a particular point. Computer technology is not used in mathematics classrooms at any level of education.

In the second semester of the first year, the students do a calculus course, Calculus I. The content covered include: the formal and the informal definitions of limit (content not covered by the subjects in this study), techniques of calculating limits in the context of functions, differentiation and integration techniques (see Appendix C for course outline). Students register for the Calculus I course even if they did not do well in the pre-calculus course. However, to continue into the second year of study in mathematics both courses must be passed. In the second year of study, students register for Calculus II. In this course the idea of limit is covered in the contexts of sequences and series (see Appendix D for course outline). There are however some problems that students encounter in learning calculus at undergraduate level. The next section discusses the nature of the problem.

Statement of the problem

Regardless of Lesotho's intention to produce more qualified science, mathematics and technology personnel (Education Sector Development Plan, 1992), mathematics, and in particular calculus, causes problems to mathematics students. The problems are visible especially in year 1 and year 2 of their studies.

On average about 50% of the first year science students pass both pre-calculus and calculus courses (see Table 1.1). Hence, they are the only group that could continue with mathematics at least up to their second year of study. The pre-calculus course has about 67% success rate on average and Calculus I has about 60%. As Table 1.2 shows, in the second year of study the success rate for Calculus II is also 60% on average.

Table 1.1 First year science students' performance at NUL for the academic years 1996/1997 to 1999/2000

Academic year	Total number of students	Number of passes in Pre-Calculus	Percentage number of passes in Pre-Calculus	Number of passes in Calculus I	Percentage number of passes in Calculus I	Number of passes in both Pre-Calculus and Calculus I	Percentage number of passes in both Pre-Calculus and Calculus I
1996/97	126	88	70	77	61	64	51
1997/98	138	75	54	82	59	66	48
1998/99	132	94	71	87	66	78	59
1999/00	183	133	73	101	55	97	53
Average pass rate			67		60		53

Table 1.2 Second year science students' performance at NUL for the academic years 1997/1998 to 2000/2001

Academic year	Total number of students	Number of passes in Calculus II	Percentage number of passes in Calculus II
1997/98	41	23	56
1998/99	33	25	76
1999/00	29	13	45
2000/01	26	20	77

Average pass rate			63

On average 23% of the initial science intake do mathematics in their second year of study, which is quite a disturbing situation for the country. Since NUL is the only university in Lesotho, its degree of success or failure has a large impact on the socio-economic development of the country. As Ogunniyi (1999, p. 1) observes:

> To remain competitive in the present era of globalization and marketization would require a radical and paradigmatic shift in the way we (educators) tackle the problem of inadequate SMT (Science, Technology and Mathematics) personnel.

Lesotho is aware that the need of producing more SMT personnel is not yet met. Its first Science and Technology policy document reads: "The supply and availability of Science and Technology have been inadequate over the years, in both number and disciplines, to meet the country's growth needs." (Ministry of Natural Resources, 2002, p. 21).

There are quite a number of obstacles that students may encounter in coming to understand some mathematical concepts. These include ontogenetic obstacles, cognitive obstacles, didactical obstacles, and epistemological obstacles. This study concentrates on epistemological obstacles.

Rationale for the study

Limit is a basic idea in calculus. "Without limits calculus simply does not exist. Every single notion of calculus is a limit in one sense or another." (Salas & Hille, 1990, p. 47). Instantaneous velocity is the limit of average velocities; the slope of a tangent line to a curve is the limit of the slope of secant lines; an infinite series is the limit of a finite sum; the area of a circle is the limit of areas of inscribed polygon as the number of sides increase infinitely. In the formal teaching of calculus the stated limits are obtained by methods of differentiation and integration, which the fundamental theorem of calculus refers to as reverse processes.

Without proper grasp of the limit concept, a very important branch of mathematics known as analysis would also not exist. At an elementary level, analysis deals with the notions of real number, function, limits of numerical sequences and functions, and continuity. At an advanced level these extend to analysis of several variables, complex analysis, and functional analysis (Artigue, 1991). Analysis can also be referred to as a study of infinite processes (Hollingdale, 1989; Haggarty, 1993). The implication of this description is that the concepts of infinitely large and infinitely small are also important in analysis courses.

As shown earlier in the chapter, despite the importance of the idea of limit in calculus, students continue to hold incomplete and alternative conceptions of it even after careful instruction. However, "this does not prevent them from working out exercises, solving problems and succeeding in their examinations." (Cornu, 1991, p. 154).

From the concerns and issues raised in the preceding paragraphs, carrying out research by studying how mathematics students at undergraduate level conceptualise the idea of limit may help in suggesting ways to alleviate the problem. The knowledge derived from this research study is a contribution to the mathematics education literature. Hence, it may be useful not only to NUL mathematics classrooms but to other students at undergraduate level elsewhere. Since the study has the potential to impact on the socio-economic status of the country in the long run, it was important to pursue it. Some specific questions were asked and addressed in investigating the nature of the problem.

Research questions

As explained in the introductory part of the chapter, most undergraduate mathematics students at NUL meet calculus for the first time at the university. This involves a difficult transition from a position where the concepts have an intuitive basis founded on experience to one where the concepts are specified by formal definitions and their properties constructed through logical deductions (Cornu, 1991; Orton, 1992; Tall, 1992). During this period of transition and long after, a variety of mental conflicts occur as new knowledge interacts with the old. This process requires the reconstruction of existing schema. During the reconstruction of schema, the incomplete accommodation itself can act as an obstacle to learning (Herscovics, 1989; Tall, 1991a; Tall, 1992).

While the major purpose of the study was to investigate the epistemological obstacles that mathematics students at undergraduate level encounter in coming to understand the limit concept, the role of language and symbolism in understanding the limit concept also formed part of the investigation. This is because communication in the mathematics classroom takes place by using language and symbols. The investigation was done by making an attempt to answer some questions on issues related to knowledge acquisition. These are:

1. What epistemological obstacles do mathematics students at undergraduate level encounter in coming to understand the limit concept?
2. What is the role of language and symbolism in understanding the limit concept?

Questions of this nature have been investigated by mathematicians and mathematics educators elsewhere, especially in case studies using different methods and analytic perspectives in interpreting the data collected. Some theoretical perspectives that were used include theories of understanding (Sierpinska, 1990), concept image and concept definition (Tall & Vinner, 1981; Aspinwall & Miller, 2001), process-object transition (Dubinsky, 1991; Tall, 1991b) and actions-process-object-schema, APOS theory (Cottrill *et al*., 1996; Juter, 2003a, 2003b, 2004, 2005).

While this study has similarities with some of the mentioned studies, it also has some differences. A study of this kind has not been conducted in Lesotho before. In the study by Cottrill *et al*. cooperative learning was used. Computers were also used to assist students to make the mental constructions proposed by the APOS framework. In the study by Sierpinska an interactive lecture method rather than computers were used. Through experience as a lecturer and having visited some lecture classrooms at NUL, the lecture method typically used is not very interactive. Only a few students may ask questions in a lecture. Most of the time, the talking is done by the lecturer. Knowing whether computers are used or not in a learning environment may shed light on why in some cases the questions may be asked the way they are. In a computer assisted environment difficult functions may be handled within a short space of time while it may not necessarily be the case in a non-computer assisted environment. This practice does, however, have its own problems.

In handling difficult functions if, for example, a student is asked to find limit values for certain functions, graphs of this function can be drawn by the computer and the limit values be found using the graphs. But the question is: How sure are we that the students are not missing some of the mental constructions that could be useful in promoting understanding if computers are used? This is because the students will not be experiencing the mental constructions first hand. But the instructors believe that students will be in a position to make the same mental constructions as the computers (Cottrill *et al*., 1996). Surely each mode of operation has its pros and cons.

The sample for the study by Sierpinska was a group of humanities students while in this study the group of science students was investigated. In the context of this study English as a second language for the subjects was the medium of instruction. Students had thus to cope with rich ideas in mathematics where all the teaching is conducted in their second language. In places like South Africa and Sweden where Aspinwall and Miller and Juter respectively conducted their studies, English is also not the first language. Because of diversity of native languages in South Africa, English is mostly used as a medium of instruction. Knowing the subjects' status position in language proficiency is important in this study because language may affect the way the subjects respond to the questions. Some epistemological obstacles encountered may be traced back to language rather than the subject matter knowledge. Language issues are also important because the second research question is concerned with the role of language in learning mathematical concepts, in this case the limit concept.

Other differences among the subjects in the different studies that could be mentioned are: the socio-cultural, the historical, and school cultural background. These differences constitute both the context of the study and the environment within which the learners interact during their time of study. In a learning environment, learners acquire knowledge through interacting with each other. Hence, they may share certain ways of knowing. The school history and the school culture are equally important. It is important to know the kind of progression that the school has attained in academic performance over time so that problems that students encounter in learning may be related to them. It is also important to know how the teachers interact with students as this has an impact on students' performance in learning. In stating the problem for this study, for example, as part of the history of the FOST, the quantitative data of students' performance over the years in mathematics is provided. This does not only confirm that there is a high failure rate at undergraduate level at the university but it also provides some context within which the problems encountered could be understood or interpreted.

In this chapter the reasons that led to pursuing the study have been discussed. An issue of how the Lesotho classrooms compare with others elsewhere has also been highlighted. In Chapter 2, some terminology constituting the topic of the study is explored with the intention to come up with the operational definitions for the key concepts for the study. Chapter 3 discusses the theories that will be used in analyzing and interpreting the data for the study. Chapter 4 discusses how the research instruments were constructed and how they have been used in pursuing the study. Issues concerned with the credibility of the research results are also discussed. Possible limitations of the study are given. Discussions of findings of the collected data appear in Chapters 5, 6 and 7. Chapter 8 begins by giving some reflective thoughts of the researcher that have resulted from pursuing the study. Then conclusions are drawn and possible implications of the research findings of the study are given.

Related literature review

In this chapter a body of literature is reviewed with the purpose of studying what mathematicians and mathematics educators have already done in investigating epistemological obstacles in developing and understanding the limit concept at undergraduate or college level. The review covers important issues from work with historical, theoretical, and empirical perspectives. The role of language and symbolism in learning calculus also forms part of the review. The discussion in this chapter will cover:

2.1 Epistemological obstacles;
2.2 Understanding in mathematics;
2.3 The role of language and symbolism in understanding the limit concept;
2.4 Epistemological obstacles identified in the historical development of the limit concept; and
2.5 Epistemological obstacles in understanding the limit concept in education.

Epistemological obstacles

In this section a brief historical background to the use of the term 'epistemological obstacle' is provided. The relations between epistemological obstacles and obstacles such as cognitive obstacles, didactical obstacles, and ontogenetic obstacles are made. At the end of the section an operational definition of the term 'epistemological obstacle' for this study is given.

The idea of epistemological obstacle was first introduced within the context of the development of scientific knowledge by Bachelard (Herscovics, 1989; Cornu, 1991; Brousseau, 1997). Bringing this idea into mathematics became possible and necessary through the development of the Theory of Didactical Situations, which had the concept 'informational leap' (Brousseau, 1997, p. 98). The concept 'informational leap' has close connection to the idea of epistemological obstacles because they both share the view that progression in knowledge acquisition is by leaps and it is not smooth.
In relating the idea of 'epistemological obstacle' to the process of knowledge acquisition Bachelard says:

> When one looks for the psychological conditions of scientific progress, one is soon convinced that it is in terms of obstacles that the problem of scientific knowledge must be raised. The question here is not that of considering external obstacles, such as the complexity and transience of phenomena, or to incriminate the weakness of the senses and of human spirit; it is in the very act of knowing,

> intimately, that sluggishness and confusion occur by the kind of functional necessity. It is there that we will point out causes of stagnation and even regression; it is there that we will reveal causes of inertia which we will call epistemological obstacles. (Hercovics, 1989, p.61).

What one gets from this translation is that, the development of scientific knowledge has experienced some hurdles or periods of slow development. The causes of the delay in the development of the scientific knowledge are given the term 'epistemological obstacles'. These obstacles are unavoidable and they are connected to the important pieces of knowledge to be acquired. Kinds of epistemological obstacles stated by Hercovics identified from the work of Bachelard include:

- The tendency to rely on deceptive intuitive experiences;
- The tendency to generalise; and
- The obstacles caused by natural language. (ibid.).

Examples reflecting how the listed aspects may cause sluggishness in the knowledge to be acquired are discussed next.

Suppose the question asked is 'Is 0.999… less than or equal to one?' An answer to this question based on intuition is likely to be that 0.999… is less than one. Because the 9's repeat and the number will never reach one. But we know that 0.333… is equal to $\frac{1}{3}$, and 0.333… x 3 is equal to 0.999…. It can therefore be concluded that 0.999… is equal to one. This is because 0.333… x 3 = $\frac{1}{3}$ x 3 = 1. Hence, they are equal. In this case intuition will have misguided the choice of answer in the first instance. But through logical deduction such a conception is falsified.

The tendency to generalise as an epistemological obstacle can be explained through the *generic extension principle* from the work of Tall (1991a, p. 10):

> ***If an individual works in a restricted context in which all the examples considered have a certain property, then, in the absence of counter-examples, the mind assumes the known properties to be implicit in other contexts.***

Two examples that may reflect the existence of this type of obstacle given by Tall are:

- A case where convergent sequences introduced to beginning students are described by the simple formula such as $\frac{1}{n}$, which tends to the limit 0, but the terms never equal the limit. It is also monotone with all positive terms. In the absence of counter examples the students will believe that the behaviour displayed is always true for other sequences.
- Another generalisation that could be made is when finding the limit of the sequence such as: 0.9, 0.99, 0.999, 0.9999, …, which has all terms less than one. The students may conclude that since the terms of the sequence are less than one , the limit value must also be less than one. That is, the limit value has to behave the same way as the terms of the sequence.

Since some technical contexts share some of their terminology with natural language, it is possible that when such words are used in technical contexts a learner may retrieve a different meaning to the intended. For example, in everyday life when the word 'limit' is used it may refer to "a boundary", "an endpoint", "a maximum", (Cornu, 1991, p. 155), but in a mathematical context this word is used with a unique meaning. In section 2.3, examples showing how the subjects confused the everyday meaning of the word 'limit' with its meaning in a technical context are given from the work of Frid (2004).

Cornu (1991) differentiates between four types of obstacles: cognitive obstacles, genetic and psychological obstacles, didactical obstacles, and epistemological obstacles. According to Cornu, cognitive obstacles occur when students encounter difficulties in the learning process. Genetic and psychological obstacles occur as a result of personal development of the student. Didactical obstacles occur because of the nature of the teaching and the teacher, and epistemological obstacles occur because of the nature of the mathematical concepts themselves.

These descriptions by Cornu seem to give an impression that there is a clear distinction between these kinds of obstacles. Knowledge acquisition takes place in a very complex system of interaction. One such subsystem that could be mentioned consists of the teacher, the student, and the knowledge system (Brousseau, 1997). When a learner experiences an obstacle in learning, how are we to apportion the blame on the system of interaction? Is it not possible for a learner to experience an obstacle in the process of learning due to the nature of teaching and the teacher? Is it not possible for a learner to experience an obstacle in learning because of the nature of the subject matter? Is it not possible for a learner to experience obstacles in learning because of the genetic and personal development? These are some of the questions that could be asked. Support for such concerns comes from the work of Brousseau (1997). Brousseau suggests that cognitive obstacles may be ontogenetic, didactical, and epistemological. This shows that there is an overlap between these obstacles. A clear distinction between these obstacles in reality may not necessarily be simple because of the complex nature of knowledge acquisition.

Hercovics's (1989) view is that Bachelard defined the notion of epistemological obstacle in the context of the development of the scientific thinking in general and not in terms of individual learning experiences. Unlike Cornu who differentiated obstacles by how they are acquired, Hercovics differentiates obstacles by reference to context. As a matter of emphasis on his classification he writes "… just as the development of science is strewn with *epistemological obstacles,* the acquisition of conceptual schemata by the learner is strewn with cognitive obstacles" (ibid., p. 61). Thus Herscovics prefers to use the term 'cognitive obstacle' in education and the term 'epistemological obstacle' when referring to the past.

There are however quite a number of researchers who believe that epistemological obstacles only appear in part in the historical development of the mathematical concepts.

Some are experienced in educational practice today though not in the same way as they were experienced in the past. Such researchers include Sierpinska (1987), Cornu (1991) and Brousseau (1997). The very nature of concept development has made it possible for learners today to bypass some of the epistemological obstacles of the past. Cornu (1991, pp. 159–161) mentions the following as the major epistemological obstacles of the past:

- The failure to link geometry with number;
- The notion of infinitely large and infinitely small;
- The metaphysical aspect of the notion of limit; and
- A question of whether the limit is attained or not.

In the past, the Greeks knew how to find the area of any polygon by dividing it into triangles and then add the areas of the triangles. The difficulty was in finding the area of curved figures. The Greeks attempted to solve this problem by inscribing polygons in circular figures. These geometrical methods acted as epistemological obstacles to speeding up the process of finding the unifying concept of limit of number (Cornu, 1991).

The two competing ideas of infinity were the potential infinity and the actual infinity (Tall, 1992). Aristotle believed that the infinite is potential and never actual. This is the same view that Kronecker held. He referred to infinity as simply a process that never could be an actual number. Another difficulty that faced mathematicians in the past was the status of infinitely small quantities. Did they exist; were they zero or non-zero, or were they 'ghosts of departed quantities' as Berkely called them? (Hollingdale, 1989, p. 305). These conceptions took a very long time to be clarified as will be seen in section 2.4. They were an epistemological obstacle.

The question of 'whether a limit is attained or not?' has also been an issue of debate that has lasted throughout the history of the development of the limit concept (Cornu, 1991). D'Alembert's position was that the magnitude from the sequence may never exceed the magnitude it approaches. Newton did not commit on whether or not the limit value is attainable. It took mathematicians a long time to unravel this mystery. Some of these points will be elaborated further in section 2.4 when discussing the historical development of the limit concept.

Tall (1989) uses the term ‘cognitive obstacle’ for epistemological obstacle. Like Herscovics, he believes that epistemological obstacle is the term that belongs to history. He mentions the sequencing of topics and the use of simple examples as examples of cognitive obstacles. That is, they are the causes of stagnation of knowledge acquisition by individual learners. Tall suggests that some topics have to be sequenced according to the level of difficulty. For example, in school teaching fractions are treated before whole numbers. However, there is a problem with this sequencing in that having met situations where multiplication of whole numbers produces bigger numbers, a generalisation that “multiplication makes bigger” could be made (ibid., p. 88). When learners encounter multiplication with fractions smaller than one, they may experience cognitive obstacles as they meet a situation where ‘multiplication makes smaller’.

The idea of sequencing of topics as an obstacle raises a lot of questions in connection with the way some topics are sequenced in NUL calculus classrooms. As mentioned in the introductory chapter, the limit of a function is covered in the first year of study while the limit of a sequence is covered in the second year of study. Such sequencing may imply that the two concepts are disconnected or the one treated at a later stage is more complicated than the one treated earlier. Hence, when students meet the concept of the limit of a sequence in their second year of study, they are likely to deny that sequences are special types of functions as the two concepts were never treated simultaneously. Cognitive obstacles arising from this sequencing of topics may therefore be experienced.

Tall suggests that if students dwell on simple examples for a longer period of time, they are likely to encounter cognitive obstacles when moving to complex cases. The idea of treating simple cases for a long period could also be related to the idea of giving examples of similar nature of a concept. For example, if when treating the topic ‘sequences’ students are introduced to monotonic sequences for a long period of time when meeting other types of sequences such as alternating sequences, cognitive obstacles are likely to occur.

Brousseau (1997) and Cornu (1991) agree that epistemological obstacles are made evident by errors in the answers that students give in responding to selected tasks and questions. But such errors are not due to chance. In fact, they persist and resist being rejected, as they are components of the acquired piece of knowledge. They further show that problems and difficulties that students meet in learning are also good indicators of existence of epistemological obstacles because they show periods of slow development of the concept. Brousseau's view is that in pursuing research "when an error, a difficulty or a problem is identified, it should be reformulated not in terms of lack of knowledge but of knowledge, false or even incomplete" (ibid., p. 94). This view seems reasonable because some causes of stagnation are unavoidable conceptions as already shown.

Brousseau (1997) suggests that in order for an epistemological obstacle to be overcome, there must be a sufficient flow of new situations which the existing schema cannot assimilate. This will destabilise it, make it ineffective, useless, or wrong in the new context, which will necessitate reconsidering it, rejecting it or forgetting it. Sierspinska (1987) on the other hand suggests that in order to overcome epistemological obstacles a mental conflict is bound to occur and therefore a didactical situation has to be introduced.

The preceding discussion has shown that differentiating the term epistemological obstacle from the other types of obstacles is a very complicated matter. Some authors believe that there is an overlap between epistemological obstacles and other types of obstacles. Others believe that there is a clear distinction between obstacles. Besides this complexity, there is evidence that students do encounter obstacles in acquiring knowledge. In this study the term 'epistemological obstacle' will be taken to mean any causes of stagnation or inertia in the knowledge to be acquired; whether in the historical development of the concept or in educational practice today. As the purpose of learning is to achieve understanding, the next section discusses theories of understanding.

Understanding in mathematics

This section presents some perceptions of understanding from a variety of theoretical perspectives. These are compared and contrasted. The section ends up by giving the operational definition of 'understanding' for the study.

Duffin and Simpson (2000) have identified and named three components of understanding as the building, the having, and the enacting. They define the 'building understanding' as the formation of connections between internal mental structures. The 'having understanding' is said to be the state of these connections at any particular time and the 'enacting understanding' is defined as the use of the connections available in the moment to solve a problem or construct a response to a question. Thus this is the type of understanding that may be visible from students' work when responding to mathematical tasks. Duffin and Simpson also talk about the breadth and depth of understanding. They describe the breadth of understanding to be determined by the number of different possible starting points that the learner may have in solving a problem. The depth may be evidenced by the way the learners can unpack each stage of their solution in more detail by referring to more concepts.

In relating these theories of Duffin and Simpson to mathematical content an example that follows may be considered. Imagine a situation where one is given a quotient function say, $f(x) = \frac{x^3}{x^2 + 1}$, to differentiate. A learner who sees this function being represented structurally as, $f(x) = x^3 (x^2 + 1)^{-1}$, can apply either the quotient rule or the product rule in solving the task. Another equivalent form of the given function obtained by dividing the top by the bottom is $x - \frac{x}{x^2 + 1}$. A learner who is aware of all these possibilities has a breadth of understanding. This flexibility is very important in the manipulation of the syntactic structures as some structures may be easier to work with than others. A learner who sees this function to be represented structurally in one form only lacks breadth of understanding. Such a learner may even deny that the given equivalent forms of the function to be differentiated represent the same function.

The depth of understanding in this case could be determined by the learner's ability to state at each stage what is happening in mathematical terms. For example, a learner could indicate the stages at which the power rule, the product rule or the quotient rule have been applied, that is, alongside the work shown in solving the mathematical task. This demonstrates a deeper understanding than in a case where the structures will be manipulated by applying a rule with no explanations at all. Reasons for applying or doing certain procedures could also be given. A learner who instrumentally carries out manipulations is likely to be unaware of the mistakes he or she has committed.

Hiebert and Carpenter (1992, p. 67) define understanding in mathematics in terms of the way information is represented and structured as follows:

> A mathematical idea or procedure or fact is understood if it is part of the internal network. More specifically, the mathematics is understood if its mental representation is part of a network or representations. The degree of understanding is determined by the number and the strength of the connections. A mathematical idea, procedure, or fact is understood thoroughly if it is linked to existing networks with stronger and more numerous connections.

This definition has close resemblance with that of Duffin and Simpson in that it puts emphasis on connections. While Duffin and Simpson refer to the building understanding, the having understanding, and the enacting understanding, Hiebert and Carpenter refer to internal and external representations. By internal representation they mean the mental representation which is not directly accessible like the having understanding. By the external representation they mean spoken language, pictures, or physical objects which Duffin and Simpson also refer to as physical markers of enacting understanding. In building understanding Hiebert and Carpenter suggest that the mental representations are built gradually as new information is connected to the already existing networks or as new relationships are constructed between previously disconnected information.

These theories of Hiebert and Carpenter are built on the assumption that the nature of the external mathematical representations influences the nature of the internal mathematical representations. That is, the type of the external representations (e.g. symbols, pictures) that a student interacts with affects the way an idea is represented internally. Conversely, the way the students produce or generate an external representation when enacting the understanding, reveals something about the way an idea is represented internally. Though Hiebert and Carpenter do not explicitly talk about the breadth of understanding, they however, describe the depth of understanding as being determined by the strength and the number of connections in the internal network. Within this framework, one could think of the following example:

Suppose we have a numerical representation of a sequence: $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, ...\}$, and the following questions are asked:

1. What happens to the terms of the given sequence as $n \to \infty$?
2. Does the given sequence converge or diverge?
3. What is the limit of the given sequence?
4. Can the given sequence attain its limit value?

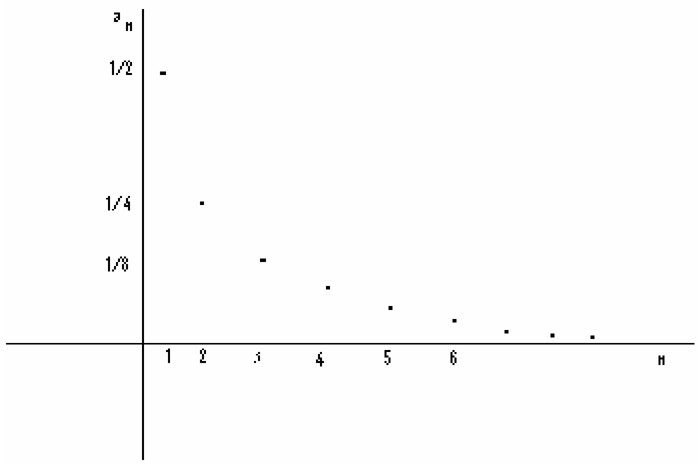
The learner who understands should be in a position to make the connections in relation to the posed questions. The learner should realise that as $n \to \infty$, the terms tend to zero, the limit value. The learner should also be aware that the terms of the given sequence converge to zero, the limit value. Hence the process of tending to and convergence in this case mean the same thing. Now in responding to the last question, one should realise that none of the terms of the sequence can ever be zero. The terms can only get close to zero but never become zero. Hence, the sequence cannot attain its limit value.

Now, if the relations are to be made across representations, the subjects could be asked questions with regard to the same sequence in different modes of representation:

Consider using the algebraic and the geometrical modes of the same sequence as shown:

(a) $a_n = \frac{1}{2^n}$, and

(b)



The questions that could be asked, to find out if relations are formed within representations, could be:

1. Given that n = 1, 2, 3, 4, … write the values of a_n in (a).
2. Write the coordinates of the points in diagram (b).
3. What do you notice about a_n values in (a) and those of the second ordinates in (b)?
4. Is there any relationship between these values and the terms of the numerical sequence $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, ...\}$? If so, what is it?
5. Find the limit values of each of the given sequences.
6. What can you say about these limit values? Are they the same? Why do you think so?
7. What differences if any do you notice about the given sequences?

Understanding will be achieved by a subject who realises that the numerical form of the given sequences is $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, ...\}$. They should also realise that the same sequence is represented. Hence, there is only one limit value. The difference should be seen in the use of different modes of representation. If relations between these representations are not made then understanding is not achieved. This could be indicative if different limit values are obtained.

Pirie and Kieren (1989, p. 8) describe understanding as a dynamic process that is recursive in nature by saying:

> It is a recursive phenomenon and recursion is seen to occur when thinking moves between levels of sophistication. Indeed each level of understanding is contained within succeeding levels.

This description suggests that the acquisition of new knowledge is dependent on prior knowledge and that the higher degree of understanding subsumes the preceding levels. Hence, it can be deduced that understanding is achieved in degrees. Thus for Skemp (1986) and Hiebert and Carpenter (1992) understanding is not an 'all-or-nothing state' or 'an all or none state'. The learner who is able to make relations within one mode of representation of an idea is at the lower level of understanding than the one who can make relations across many representations of the same idea.

Skemp (1976) differentiates between two kinds of understanding, the instrumental and the relational. Instrumental understanding is described as knowing 'rules without reasons' and relational understanding is 'knowing both what to do and why' (ibid. p. 20). In the context of limit one could say, knowing that the limit of the function $f(x) = \frac{1}{x}$ as x tends to ∞ is zero without knowing why this is so, is an instrumental understanding. This could happen when such a function has been used as example over and over again. If one knows that the limit is zero because $f(x)$ values tend to zero as x values tend to infinity, then this will be a piece of knowledge that is achieved through relational understanding.

Instrumental understanding as Skemp observes, is predicted from the difficulty of accommodating or restructuring the existing schemas. One advantage that instrumental understanding has over relational understanding is that its rewards may be more immediate. One may get the correct answer very quickly by using the instrumental methods. Relationally learnt material is not only better learned but better retained. It is also adaptable to new tasks. Sometimes instrumental understanding may be used to aid relational understanding. If a procedure is well performed for example, a correct answer will be obtained and the process of making appropriate connections between concepts within the same network may be facilitated in this regard.

Byers (1980, pp. 5–6) gives the following description of understanding:

> It is impossible to understand a piece of mathematics in the absence of pre-requisite knowledge. Moreover the understanding of mathematics deepens with the acquisition of new mathematical knowledge … understanding involves availability for ready retrieval. What is required, however, is not so much the retrieval of isolated bits of information as the availability of organized relevant knowledge.

For Byers, knowledge of mathematics is a necessary but not a sufficient condition to achieve understanding. In order for understanding to occur the pieces of knowledge have to be connected. The idea of connectedness seems to cut across all the discussed theories of understanding in this section. As in the case of Pirie and Kieren it could be argued that Byers also sees the different levels of understanding to be contained in another. This emanates from Byer's view that new knowledge is built on the old. This view also resonates with Ausubel's (1985, p. 82) theory of meaningful learning:

> If we had to reduce all the educational psychology to just a single principle, we would say this: Find out what the learner already knows and teach him or her accordingly.

Thus in building understanding, the learner should have the relevant prior knowledge. In order to understand the concept of the limit of a sequence, for example, learners should have knowledge of sequences in different modes of representation. Other things that need to be known are the position of the terms of a sequence, the terms of the sequence and the associated symbolism, the concept of convergence/divergence in relation to any given sequence, and possibly translating a sequence from one form of representation to another.

Sierpinska (1990, p. 28) relates understanding and epistemological obstacles as follows:

> We know things in a certain way. But the moment we discover there is something wrong with this knowledge (i.e. become aware of an epistemological obstacle), we understand something and we start knowing in a new way. … In many cases overcoming an epistemological obstacle and understanding are just two ways of speaking about the same thing. The first is "negative" and the second is "positive" …. Epistemological obstacles

> look backwards, focusing attention on what was wrong, insufficient, in our way of knowing. Understanding looks forward to the new ways of knowing.

As already discussed, epistemological obstacles are not necessarily a result of a wrong way of knowing. Sometimes the piece of knowledge may be appropriate in one context but not the other. Sometimes the piece of knowledge may not be appropriate in any given context. Though Sierpinska refers to epistemological obstacles as negative, as pointed out by the work of Bachelard, epistemological obstacles do have a positive attribute in that they are connected to the important pieces of the knowledge to be acquired. In building up knowledge, we have to identify them and overcome them in order to make progress. Even in the history of the development of concepts, mathematicians have achieved the present state of understanding of some mathematical concepts through encountering epistemological obstacles and overcoming them.

Since students' knowledge is determined from enacted understanding through the use of words, pictures and symbols, in this study the model of understanding that will be used is that by Duffin and Simpson. This model encompasses also the theories of Hiebert and Carpenter. These theorists have explicitly talked about external representations of knowledge or understanding by referring to language and symbols. It would therefore be necessary to look at the role of language and symbolism in learning mathematics.

The role of language and symbolism in understanding the limit concept

This section discusses the role of language and symbolism in understanding the limit concept. The discussion will be approached from both the theoretical and the empirical perspectives. Epistemological obstacles that students are likely to encounter due to language and symbolism in understanding the limit concept are also discussed.

Language is a system of communication that consists of a set of sounds and written symbols used by people for talking or writing (Collins Dictionary, 1992). In mathematical context, communication may also be described as the mechanism used by teachers and learners alike in an attempt to express their mathematical understandings

(Pirie, 1988). Special features of mathematical language include vocabulary, syntax, and symbols (Grifiths & Clyne, 1994). In calculus some vocabulary that has to be known includes limit, limiting process, infinity, infinitesimal, approaches, tends to, differentiate, and integrate.

A symbol can be described as a word or mark that stands for something but in no way resembles that thing (Resnick & Ford, 1984). There are however, some symbols that resemble the concepts they represent. For example, the symbol for a circle 'O' has close resemblance with the idea it represents. Skemp (1986) defines a *symbol* as a sound or something visible mentally connected to the idea. That idea according to Skemp is the meaning of the symbol. Skemp differentiates between two types of symbols, *verbal* and *visual*. Verbal symbols are spoken or written words. These include symbols such as 'л' for 'pi'. Visual symbols are diagrams of all kinds. The definition used by Skemp is the one that will be used in this study.

Mathematics has a set of rules that govern strings or collection of symbols in a technical sense to, constitute a valid mathematical sentence or expression. This is called syntax or the syntactic structure of the symbol(s). There are however some conflicts within these rules. For example, there is the conflict between symbolic structure and mathematical language, the conflict between symbolic structure and ordinary language, and the conflict between the symbolic 'surface structure' and the 'deep structure'. The meanings conveyed by the surface or the syntactic structures are called deep structures (Orton, 1992).

The following pairs of expressions, for example, have the same surface structures but different deep structures: $3x$ and Δx, $\frac{\delta y}{\delta x}$ and $\frac{dy}{dx}$. $3x$ means 'multiply 3 by x whereas Δx means 'change in x'; Δ and x are observed as one and therefore cannot be separated. $\frac{\delta y}{\delta x}$ is a quotient, which means 'divide δy by δx', but $\frac{dy}{dx}$ has to be observed as one. It is a command that mathematically tells us either to find the derivative of the expression or to differentiate.

Ausubel (1985) mentions the major functions of symbols as representation and communication. Zaskis and Liljedahl (2004) also see the major functions of symbols as representation and communication. They differentiate between two kinds of representation, transparent representation and opaque representation. In representing 784 as 28^2 makes the property of 784 being a perfect square transparent and the property of divisibility by 98 opaque. In their study on the role of understanding prime numbers through representation they found out that the major obstacle to the understanding of the concept of number by students was due to the use of opaque representation. Thus representation may act as an epistemological obstacle to the understanding of mathematical concepts.

Zaskis and Liljedahl (2004) see representation as the major tool for manipulation and communication. They suggest that having representation in hand allows learners to detach themselves from the meaning of the representation and operate on the symbols alone. This situation makes the manipulations automatic. The learners may return to the interpretation of the result of the symbolic manipulation at a later stage. Similar views are given by Mason (1987). Mason suggests that in using symbols the attention is easily given to the syntactic structure rather than the semantic deep structure. An example that one would give in the context of calculus is when given the mathematical task: Find $\frac{d}{dx}(x^2)$ at $x = 2$. When seeing this kind of problem both the learner and the teacher alike would quickly think of the power rule and get the derivative as $2x$. Substitute 2 in $2x$ and get 4 as the answer. This can be worked out in at most one minute. But when asked to relate the answer to the meaning, the semantic or deeper structure, one might find it difficult to see or even think that they are actually talking about the gradient of the tangent line to the curve $y = x^2$ at $x = 2$ being 4. In the historical development of the limit concept mathematicians like Fermat, Newton and Leibniz also concentrated on the manipulation of the surface symbolic structure rather than the deeper structures. This point will be revisited in section 2.4.

A study by Frid (2004) conducted on calculus students at one university and two colleges, found out that in cases where students were able to manipulate or perform operations with symbols, they did not use symbols as representing concepts. When students were interviewed on how they see the role of symbols in understanding calculus concepts, this is what three subjects, Richard, Ellen, and Cindy, had to say:

Richard: They are just symbols I move around according to a rule. They don't really mean anything. … It doesn't have a meaning. . It seems like it's stupid notation. Why don't they have the notation that says what it is? (ibid., p. 12).

Ellen: You have a variable *x* and *y*. Why do you have a *d* in front of it? Or why do you have a little slash thing on it having the derivative? Like what does that mean? (ibid.).

Cindy: I guess you have to have them but I just get really, really confused. There's so much. And I don't think there's enough attention given to making us understand all the symbols. (ibid.).

These extracts do show that when students use symbols they do sometimes separate them from the concepts that they represent. These findings confirm the views of Mason (1987) that when students are engaged in mathematical tasks they focus on the manipulations rather than concepts. The last statement by Richard: "why don't they have the notation that says what it is?" suggests that Richard sees the symbolism used in calculus as opaque because it hides the meaning of the concepts it represents. In the same study by Frid (2004), for example, it was found out that some subjects committed errors in applying the differentiation rules. In differentiating a composite function, the subjects did not apply the chain rule appropriately as they could not recognize a composite function. The subjects could not differentiate an inner function from an outer function.

Janvier (1987) mentions the major role of symbolism as that of a translation process. He describes the translation process as the psychological processes in going from one mode of representation to another. He gives a list of some translation processes as: interpretation, computing, sketching, and parameter recognition. Zaskis and Liljedahl (2004) see the ability to move between various representations of the same concept as an indication of conceptual understanding. This is because meaning is brought about as these representations are connected to the ideas they represent.

One of the major communication problems that mathematics faces is that it shares some technical terms with ordinary language (Pirie, 1988; Orton, 1992; Tall, 1992). Some vocabulary that mathematics shares with ordinary language include terms such as limit, range, differentiate, integrate, sequence, converge, and the phrase 'tends towards'. Cornu (1991) lists some spontaneous conceptions of the phrase 'tends towards' and the term 'limit'. The spontaneous conceptions of the phrase 'tends towards' are given as:

- To approach (eventually staying away from it);
- To approach … without reaching it;
- To approach … just reaching it; and
- To resemble (without any variation, such as "this blue tends towards violet". (ibid., p.154).

The spontaneous models of the term limit are listed as:

- An impassible limit which is reachable;
- An impassible limit which is not possible to reach;
- A point which one approaches, without reaching it;

- A point which one approaches and reaches;
- A higher (or lower) limit;
- A maximum or minimum;
- An interval;
- That which comes 'immediately after' what can be attained;
- A constraint, a ban, a rule; and
- The end, the finish. (ibid., p. 155).

In responding to mathematical tasks concerning limits, the above conceptions are likely to form part of the learners' retrieved schema. Hence, they may act as epistemological obstacles. There are some research findings that confirm this claim. In Frid's (2004) study, when students were asked what their understanding of limit was, this is what two subjects had to say:

> Daniel: "The limit for myself represents a barrier or endpoint at which something is possible. For example, a swimmer would only be able to swim one mile because that is the limit of his or her endurance. Similarly in math, though more complex, a limit represents a maximum or minimum possibility". (ibid., p. 18).
>
> Sally: Something that a number approaches, but it will never reach. Or something it can't cross, like a border. Like you can't ever quite get to it. (p. 18).

These extracts show how everyday meaning of words interfere with their mathematical meaning. The words 'endpoint', 'maximum', 'minimum' and the phrase 'approaches but will never reach' appear in a listing provided by Cornu as spontaneous models of the word 'limit'. Other words that have been used by the subjects are barrier and border. These words are also equivalent forms of the concept of limit in everyday life. Thus everyday language acts as epistemological obstacle here because it may delay the progress of acquiring the limit concept.

In this section one has discussed the role of language and symbolism in learning mathematical concepts. The examples within the concept of limit were also given. The next section discusses epistemological obstacles in history that have led to the slow development of the limit concept. This is because the epistemological obstacles that students encounter in the educational practice today in learning calculus might be understood better if put in context of the epistemological obstacles of the past.

Epistemological obstacles in the historical development of the limit concept

This section begins by stating some problems that inspired the creation of calculus. These are followed by a discussion on the causes or the reasons for the stagnation in the development of the limit concept. The last part of the discussion shows how the limit concept became a solution to the stated problems.

The calculus was created in the 17^{th} century mainly to solve the motion problems. They are to find:

- The instantaneous rates of change;
- The tangent and normal to a curve at any point on it;
- The maximum and minimum values of a function;
- The length of curves, areas enclosed by curves, centre of gravity of such areas and volumes (Boyer, 1949; Muir, 1961; Kline, 1972; Hollingdale, 1989; Cornu, 1991); and
- The sum and convergence of a series. (Cornu, 1991).

All the stated concepts are related to the idea of limit in one way or another. For example, the instantaneous velocity is the limit of the average rates of change. The gradients of the tangent lines to the curve are the limits of the gradients of the secant lines and the gradients of the tangent lines to the curve at the minimum and the maximum points are both zero. The area of the circle, for example, is the limit of the inscribed polygon whose number of sides increase indefinitely. The sum of a series is obtained by the partial sums converging to a unique value called the limit.

The discussions will be divided into three subsections, the developments made by the predecessors of Newton and Leibniz, the contribution to developments by Newton and Leibniz, and the development made by the successors of Newton and Leibniz. This is because some important work in the development of calculus is dedicated to these two men.

The development made by the predecessors of Newton and Leibniz

The contributions made in the development of the limit concept, involve very many mathematicians (Boyer, 1949). The discussion in this section will take into consideration the contributions made by the Greeks and the contributions made by Fermat. The choices made are mainly to provide some focus for the discussion.

The Greeks (5th century B. C.)

During the 5th Century B.C. the Greek philosopher, Zeno, baffled his colleagues with paradoxes of motion (Muir, 1961). One paradox concerns the race between the Achilles and the tortoise that has been given a head start. Zeno argues that Achilles cannot overtake the tortoise because he must always reach the point the tortoise has passed. Thus Achilles will keep on approaching the tortoise but he will never reach the tortoise. This perception reflects an incoherent way in which Zeno thought about motion. The distance covered by a body in motion depends upon speed and time taken. Hence, the Achilles would overtake the tortoise.

The other paradox communicated to us by Aristotle reads: "A man standing in a room cannot walk to the wall. In order to do so, he would have to go half the distance, then half the remaining distance, and then half of what still remains. The process always continues and can never be ended" (Stewart, 1999, p.7). This can be represented by a series: ½ + ¼ +…+ $\frac{1}{2^n}$ + …, interpreted as, in order to go a finite length one must cover an infinite number of points and so must get to the end of something that has no end (Kline, 1972). But there is no last point in the infinity of points (Muir, 1961). Hence, this is logically not possible.

The third paradox concerned the arrow flying through the air. The problem was that at an instant the arrow has to be at one place. Hence, it will not be moving as it will be at rest. But in reality the arrow still continues to move while in air. This reasoning contradicted reality. Zeno himself could not solve problems related to his paradoxes as they were said to be incomprehensible.

Since the Greek mathematicians did not manage to solve the paradoxes of Zeno either, they burned the concept 'infinity' from mathematics as it was believed to be troublesome (Boyer, 1949; Muir, 1961, Hollingdale, 1989). So, the Greeks got stuck to their finite, static, unmoving geometric figures (Muir, 1961). As mentioned in chapter 2, the Greeks for a long time did not know how to find the area of curved surfaces. In order to find the area of curved figures they used the method of exhaustion. This was achieved by inscribing a polygon in a circle and circumscribed it about a circle. The sides of the inscribed or circumscribed polygons were increased indefinitely to approach the area of a circle as much as desired, but could not coincide with the circle. At present we can interpret this to be that the area of the circle is the limit of the area of the inscribed polygon whose sides increased indefinitely. But the Greeks did not have the concept of limit by then. This method was used to compare the areas of figures in terms of one figure being less than or greater than the other. The method was wholly geometrical. The success of using this geometrical method was an epistemological obstacle as it delayed the passage to the idea of numerical limit (Cornu, 1991). Successful as it was, it was a very cumbersome method (Boyer, 1949). Hence, some alternative methods had to be sought.

Fermat (1601 – 1677)

Fermat was the first to tackle the problem of tangents systematically (Kleiner, 2001, p.140). In the 1630's he devised a method of finding tangents to any polynomial curve (Kline, 1972; Kleiner, 2001). His method was as follows: If P is the point on the curve $y = x^2$, with the coordinates (x, x^2), the point in the neighbourhood of P will have the coordinates $(x,(x+e)^2)$. Say we want to find the tangent to the parabola at some point P, we can do as follows:

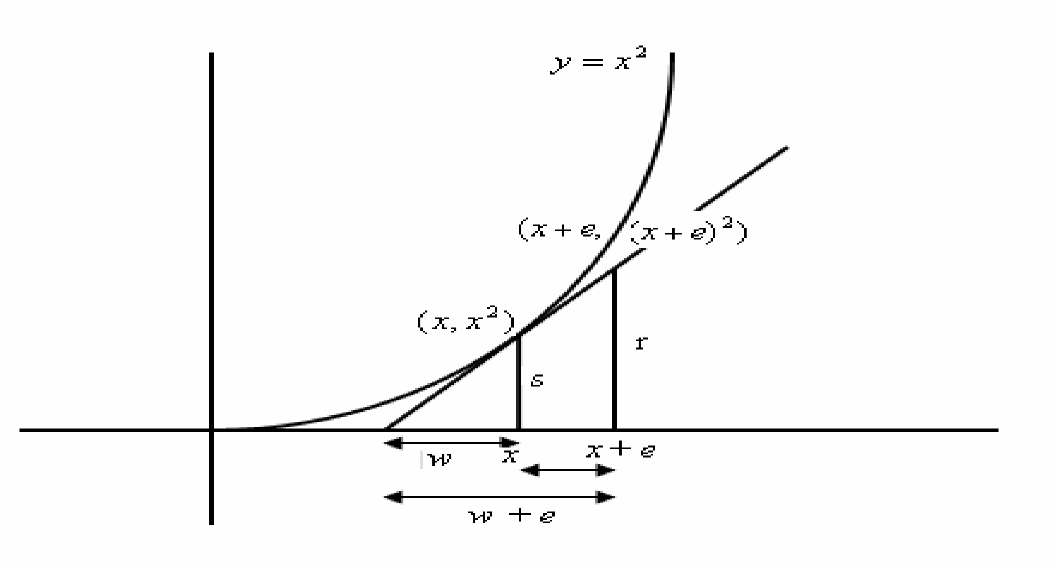


Figure 2.1 Finding the gradient of the tangent line

Since $s = x^2$ and $r \approx (x+e)^2$ by similarity of triangles we have $\frac{s}{w} = \frac{r}{w+e}$ substituting x^2 for s and $(x+e)^2$ for r yields $\frac{x^2}{w} \approx \frac{(x+e)^2}{w+e}$. When multiplying both sides by $w(w+e)$ we get $wx^2 + ex^2 \approx w(x^2 + 2xe + e^2)$. Removing brackets yields $wx^2 + ex^2 \approx wx^2 + 2wxe + we^2$. A further simplification produces $ex^2 = 2wxe + we^2$. Continuing the simplification by factoring w we get $w \approx \frac{ex^2}{2xe + e^2}$. Dividing by e we get $w \approx \frac{x^2}{2x+e}$. From this $\frac{x^2}{w} \approx 2x + e$. Now when e is deleted, the result becomes $\frac{x^2}{w} = 2x$. This is then the slope of the tangent line to the curve.

This method produces the same answer as the method of finding the derivative of the function $y = x^2$. Geometrical interpretation of the slope of the tangent line is that it is the limit of the slopes of the secant lines. In the performed manipulations, towards the end, division by e is done, which means that e is not zero. At the end, the e is deleted to get $2x$, which means that e is zero. This contradicts the rules of mathematics. We cannot have the same symbol representing two different things in the same expression. That is, an e

cannot be something and nothing at the same time. This shows that as this was done, the concentration was more on the manipulation rather than on concepts. Thus, Fermat's method was objected to by his contemporaries, in particular Descartes (Kleiner, 2001). However, Fermat's mysterious *e* embodied a crucial idea, the giving of a small increment to a variable, which needed the limit concept to succeed (Boyer, 1949; Kline, 1972; Kleiner, 2001).

In finding the minima and maxima values for polynomial curves, Fermat compared the values of $f(x)$ at a point with the value of $f(x + e)$ at a neighbourhood point. He realised that at the top of the curve the change will almost be imperceptible (Hollingdale, 1989). Meaning that as one approaches a maximum, the slope approaches zero and is zero at the maximum.

The developments made by Newton and Leibniz

This subsection discusses the contributions made by Newton and Leibniz in the growth of knowledge in developing the limit concept. This is because these two mathematicians made significant contributions in the development of the limit concept (Kline, 1972; Hollingdale, 1989; Kleiner, 2001).

Newton (1644 – 1727)

The calculus of Newton and Leibniz is a calculus of variables and equations relating the variables. It is not a calculus of functions. The function concept became a mathematical concept only in the early 18th century (Kleiner, 2001).

Newton's stages of thought in the conceptual development of the determination of the fluxion or rate of change of $y = x^n$ were reflected mainly in three stages of his publications, in *De analysi* (written in 1665 and published in 1711), *Methodus fluxion* (written in 1671 and published in 1736) and *De quadratura* (written in 1693 and published in 1704) (Hollingdale, 1989; Boyer, 1949).

In *De analysi* Newton did not use the fluxionary notation ($\dot{x}$ or $\dot{y}$), he used the idea of the infinitesimally small both geometrically and analytically (Boyer, 1949, Hollingdale, 1989). Newton employed the idea of indefinitely small rectangles or moment of area to find the quadratures (integrations) of curves. That is, for the curve $y = ax^{\frac{m}{n}}$, the area is given by $z = \frac{n}{m+n} ax^{\frac{m+n}{n}}$ (Boyer, 1949). Here Newton found the rate of change of the area, and from it found the area itself by the indefinite integral of the function representing the ordinate (Boyer, 1949). This result was obtained by: letting the moment or infinitesimal increase in the abscissa be o (following the notation of James Gregory), the new abscissa will then be $x + o$ and the augmented area $z + oy = (\frac{n}{m+n}) a(x+o)^{\frac{m+n}{n}}$ (see Figure 2.2). Apply the binomial theorem, divide throughout by o, and then neglect the terms still containing o, the result will be $y = ax^{\frac{m}{n}}$. That is, if the area is given by $z = \frac{n}{m+n} ax^{\frac{m+n}{n}}$, the curve will be $y = ax^{\frac{m}{n}}$ (Boyer, 1949). Conversely if $z = \frac{n}{m+n} ax^{\frac{m+n}{n}}$, then $y = ax^{\frac{m}{n}}$. Thus Newton at this stage recognised that integration and differentiation are reverse processes.

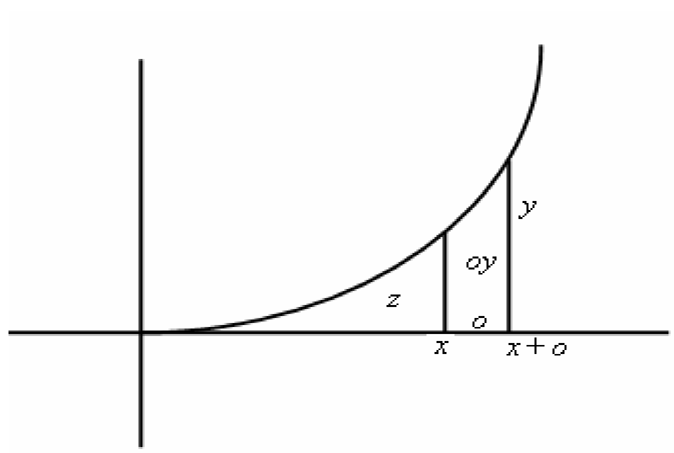


Figure 2.2 Finding the area under the curve

In finding the fluxion of $y = x^n$, in *Methodus fluxion*, his approach differed a bit from that presented in *De analysi*. His conception was that:

> If o is an infinitely small interval of time, the $\dot{x}o$ and $\dot{y}o$ will be indefinitely small increments or moments, of the flowing quantities (fluents), x and y. In $y = x^n$ one then substitutes x + $\dot{x}o$ for x and $y + \dot{y}o$ for y, expands before by using the binomial theorem, cancels the terms not containing o, and divide throughout by o and neglect all the terms that contain o…the result is $y = nx^{n-1}$. (ibid., p.194).

As in the case of Fermat the problem with this work was that of dividing by *o* in one stage and neglecting it at another. In dividing out by o meant that o was something. Neglecting *o* meant that it was nothing. And if it were zero, division by zero is not allowed in mathematics. As pointed out earlier this is a conceptual contradiction. The result $y = nx^{n-1}$ was the same as that used in *De analysi* in which the fluxions were not used. At this stage Newton felt the need for the limit concept by pointing out that the fluxions are never considered alone but in ratios (Boyer, 1949).

As in the case of *Methodus fluxion*, in *De quadratura* Newton replaced x by $x + o$ expanded $(x + o)^n$ by the binomial theorem and subtracted x^n. Instead of neglecting the terms this time or allow some of them to vanish, he formed the ratio of the changes in x to the change in x^n. He obtained the ratio 1 to $nx^{n-1} + n(\frac{n-1}{2})ox^{n-2} + ...$ then allowed o to approach zero and vanish and the resultant was, 1 to nx^{n-1} which he called the ultimate ratio instead of the limit of the ratio of the changes (Kline, 1972). At this stage one of the important elements of the derivative that was more prominent than in the earlier work was the determination of the limit of the ratio as the changes approach zero (Boyer, 1949, p. 196).

Newton's definition of limit was:

> Quantities and the ratios of quantities, which at any finite time converge continually to equality, and before the end of that time approach nearer to each other than by any given difference, become ultimately equal. (Hollingdale 1989, p. 209).

Newton appears to have been aware of the method of finding the limits of converging sequences (Kleiner, 2001). However, there are ambiguities inherent in some terminology used. For example, “What does ‘ultimately equal’ mean?” (ibid., 2001, p. 155). Other questions that could be posed in relation to the meaning of the phrases “converge continually to equality” and “become ultimately equal” are: What does convergence to equality mean when there is always a difference between the ratios of these quantities? Does ‘become ultimately equal’ mean that these ratios will attain the limit? Does it mean it will be in the neighbourhood of the limit? Answers to these questions are not easy to get from this writing.

Newton described the ultimate ratio as follows:

> By the ultimate ratio of evanescent quantities is to be understood the ratio of the quantities not before they vanish, nor afterwards, but with which they vanish…. Those ultimate ratios with which quantities vanish are not truly ratios of ultimate quantities, but limits towards which the ratios of quantities decreasing without limit do always converge; and to which they approach nearer than by any given difference, but never go beyond, nor in effect attain to, till the quantities are diminished *in infinitum*. (ibid., p. 154).

There are quite a number of questions that can be asked with regard to this description of the ultimate ratio as in the case of limit. Ratios of quantities are said to be decreasing without limit. Yet again they are said to possibly attain the limit when they diminish *in infinitum.* What does it mean to diminish *in infinitum*? Does it mean that which is something now becomes nothing? As highlighted in chapter 2 of this study, the question of whether a limit is attainable has been a problem that has lasted some centuries. It could then be argued that Newton therefore failed to resolve the difficulties posed by the limiting processes. This problem lasted until the beginning of the 19th century (Kleiner, 2001) as will be seen in the fore coming discussion. Thus the idea of ‘infinitely small’ has been an epistemological obstacle for centuries.

Leibniz (1646 – 1716)

Like Newton, Leibniz’s ideas on calculus developed gradually and his line of concept development was expressed in his writings (Kleiner, 2001). Central to his work was the concept ‘differential’. Leibniz perceived a curve as a polygon with infinitely many sides. The same conception was held by the Greeks. He used the notation *dx* to denote the distance between the successive values of *x*. Similarly, the difference between the successive values of y was called the differential of y denoted by dy. In finding the slope of the tangent to the curve, Leibniz used his famous *characteristic triangle* with infinitesimal sides, ds, dx and dy. The sides were related by the equation $(ds)^2 = (dx)^2 + (dy)^2$ (see Figure 2.3). The side ds was taken to be coincident with the tangent line to the curve. To put it in Leibniz's words:

> We have only to keep in mind that to find a tangent means to draw a line that connects two points of the curve at an infinitely small distance, or the continued side of a polygon with an infinite number of angles, which for us takes the place of the curve. This infinitely small distance can always be expressed by a known differential like *ds*. (ibid., p. 146).

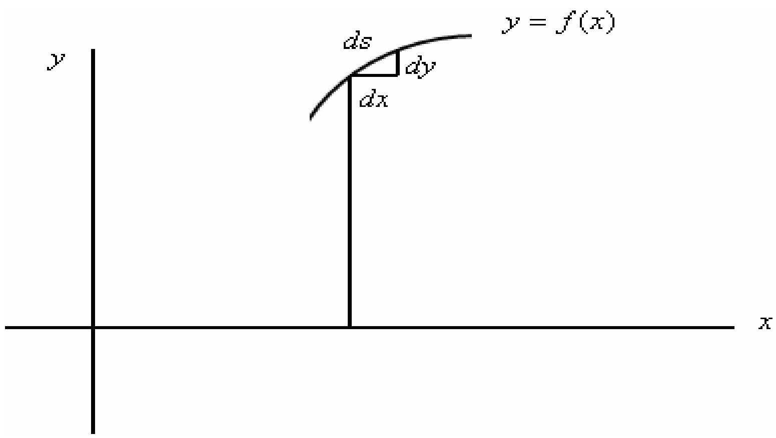


Figure 2.3 Geometric relations between differentials

The slope of the tangent to the curve at the point (x, y) was found by using the *differential quotient* $\frac{dy}{dx}$. This notation is used in calculus at present. However it does not just mean the differential quotient, but the limit of the differential quotient $\frac{\Delta y}{\Delta x}$ as Δx tends to zero. The differentials dy and dx are now replaced by Δy and Δx respectively. That is, $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, the limit of the sequence of difference quotients. Through studying the work of Barrow, Leibniz was aware of the direct and inverse problem of finding tangents. He was also certain that the inverse method was equivalent to finding areas and volumes by summations (Kline, 1972).

In 1672 Leibniz's tutor, Huygens, set him with a problem of summing up the series (Hollingdale, 1989):

$$\sum_{n=1}^{\infty} \frac{2}{n(n+1)}$$

As mentioned in the introductory part of section 2.4, the sum of a series is obtained by the partial sums converging to a unique value called the limit. We know by the partial fraction decomposition that:

$$\frac{2}{n(n+1)} = \frac{2}{n} - \frac{2}{n+1}$$

So, the sum of the infinite series s, can be written as:

$$s = (2-1) + (1-\frac{2}{3}) + (\frac{2}{3}-\frac{1}{2}) + (\frac{1}{2}-\frac{2}{5}) + ... + (\frac{2}{n}-\frac{2}{n+1}) + ... = \lim_{n\to\infty} (2-\frac{2}{n+1}) = 2 - 0 = 2$$

However, Leibniz did not employ this method. He obtained the answer 2 by applying the knowledge that:

1. Terms of this sequence are the reciprocals of the triangular numbers (1, 3, 6, 10, …). That is, $s = 1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \ldots$;
2. A sequence may be summed if each term can be expressed as a difference. Since $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, then $s = \sum_{r=1}^{\infty} \frac{2}{n(n+1)} = \lim_{n\to\infty}(2 - \frac{2}{n+1})$. Hence, the sum to infinity is 2, the limit value (Hollingdale, 1989).

The next subsection discusses the developments made by the successors of Newton and Leibniz.

The developments made by the successors of Newton and Leibniz

The successors of Newton and Leibniz to be discussed here are Euler, D'Alembert, Bolzano, Cauchy, and Weierstrass. Their choice is based on the contributions they made towards achieving rigour in the definition of the limit concept.

Euler (1707 – 1783)

As already discussed, Newton and Leibniz's calculus was a calculus of variables and not a calculus of functions. A major breakthrough was made by Euler around the mid 18th century by making the function concept the centre around which the calculus revolves (Kleiner, 2001). Unlike Leibniz who explained the quotient $\frac{dy}{dx}$ as a quotient of differentials, Euler explained it to be a quotient of zeros, $\frac{0}{0}$ (Boyer, 1949). The quotient $\frac{0}{0}$ is meaningless in the context of mathematics as the division by zero is not permissible. However, this conception matches with that of Wallis, John Bernoulli, and Fontenelle, who conceived infinitely small as the reciprocal of infinitely large. Thus they represented infinitely small as $\frac{a}{\infty} = 0$ and infinitely large as $\frac{1}{0} = \infty$ (Boyer, 1949). Even though this does not make any sense now, by then this is how they made sense in relating the concepts 'infinitely small' and 'infinitely large'. Thus these concepts were still an epistemological obstacle even at this stage.

D'Alembert (1717 – 1783)

Though Newton's definition forms the base for the present definition of limit, no one paid much attention to it (Kline, 1972). It is d'Alembert who realised that the limit concept was very central to calculus and that the derivative requires the understanding of the limit concept (Kline, 1972; Hollingdale, 1989). D'Alembert' definition reads:

> One magnitude is said to be the limit of another magnitude when the second may approach the first within any given magnitude, however small, though the second magnitude may never exceed the magnitude it approaches, so that the difference of such a quantity to its limit is unassignable …. Not only can the magnitude never exceed its limit, it cannot actually attain it either. (Hollingdale, 1989, p. 305).

D'Alembert does commit in saying that the limit is something that cannot be exceeded. Neither can it be attained. We now know that limits of continuous functions are attainable. So, this means that d'Alembert was still very far from giving the idea of limit its current interpretation.

D'Alembert's conception of an infinitesimal was that – "a quantity is something or nothing. If it is something, it has not yet vanished, if it is nothing, it has literally vanished; the supposition that there is an intermediate state between the two is a fantasy" (Hollingdale, 1989, p. 305). Up to this stage the idea of infinitely small was still an epistemological obstacle. Calculus became rigorous through the work of Bolzano, Cauchy, and Weierstrass in the 19th century (Boyer, 1949). The coming discussion concentrates on the contributions made by these three men.

Bolzano (1781 – 1848)

Unlike Euler who had explained $\frac{dy}{dx}$ as the ratio of zeros, Bolzano took this symbolism not to be interpreted as a ratio or a quotient of zeros but as a symbol for a single function (Boyer, 1949). He further said that if a function reduces to $\frac{0}{0}$, then it has no determined value at a point. However, it could have the limiting value as the function may be continuous at that point. This explanation provided by Bolzano still holds even today. The limit value may exist even where the function is not defined. But the function value does not exist at that point.

Cauchy (1789 – 1857)

Cauchy selected a few fundamental concepts namely, limit, continuity, convergence, derivative, and established that the limit concept is central to all of them (Kleiner, 2001). Cauchy's definition of the limit concept reads as follows:

> When the successive values attributed to a variable approach indefinitely a fixed value, eventually differing from it by as little as one wishes, that fixed value is called the limit of all the others. (ibid., 2001, p.161).

Though Cauchy speaks of the limit of a variable rather than the limit of a function, he however, does not commit in saying what happens when the variable approaches its limit. Will it ever reach or attain it? Can it exceed it? Cauchy's conception of the infinitesimal was:

> One says that a variable quantity becomes infinitely small when its numerical value decreases indefinitely in such a way as to converge to the limit zero. (Cornu, 1991, p. 160).

It appears as if here Cauchy means that if a sequence is generated, the terms of the generated sequence will become smaller and smaller and their numerical values will be very close to being zero. Since the decrease is indefinite, it means that they can never be zero.

The series of numbers and functions were used freely in the 17th and 18th centuries with little concern for their convergence (Kleiner, 2001). Cauchy was the first to present a systematic careful treatment of convergent series. He however burned the divergent series from analysis as they were said to be "the invention of the devil" (Kleiner, 2001, p.163). For a convergent series, Cauchy provided the following definition:

> A series converges if for increasing values of n, the sum s_n of the first n terms approaches a limit s, called the sum of the series. (Boyer 1968, p.560).

This definition still holds for limits of converging series. Cauchy also proved that the necessary and sufficient condition that an infinite series converges is that:

> For a given value of p, the magnitude of the difference between s_n and s_{n+p} tends towards zero as n increases indefinitely. (ibid., p.566).

This is true for a decreasing sequence and it is not necessarily true for a constant sequence. In considering the magnitude of the differences between terms we can generate a sequence: {½, ¼, 1/8, 1/16, 1/32, 1/64, …} the difference between the terms of this sequence as n increases indefinitely become smaller and smaller and they tend to zero. But for a sequence such as {1, 1, 1, 1, …} the difference between terms is always zero. This definition is similar to the informal definition of limit used today.

Weierstrass (1815 – 1897)

Though Bolzano was the main advocate of rigour, it is Weierstrass who was the first to formulate the static definition of a limit that is used today, the limit L of a function $f(x)$ at the point x_0 by giving the definition clarity and precision (Boyer, 1949). The definition reads:

> If, given any $\in$, however small, there is a number δ, such that for $0<\delta<\delta_0$, the absolute value of the difference $\{f(x_0 \pm \delta) - L)\}$ is less than $\in$, then L is the limit of $f(x)$ for $x = x_0$. (Hollingdale, 1989, p. 354).

At last the infinitesimals were eliminated and only real numbers, less than, and the operations of addition and subtraction were used (Hollingdale, 1989).

This discussion has shown that the present rigour has been achieved through leaps and it has not been smooth. Causes of the stagnation or epistemological obstacles have been highlighted in the discussions. The next section discusses some empirical work on the epistemological obstacles encountered in educational practice.

Epistemological obstacles in understanding the limit concept in education

Epistemological obstacles related to the limit concept, whether in the context of a function or a sequence, have not only been the problems of the past, some are experienced in the educational practice in modern mathematics (Taback, 1975, Tall & Schwarzenberger, 1978; Tall &Vinner, 1981; Orton, 1983a, 1983b; Davis & Vinner, 1986; Sierspinska, 1987; Cornu, 1991; Monaghan , 1991; William, 1991; Cottrill *et al*., 1996; Tall, 1996; White & Mitchelmore, 1996; Billings & Klanderman; 2000; Aspinwall & Miller, 2001; Juter, 2003a, 2003b, 2004, 2005). Section 2.3 focused on the role of language and symbolism in understanding mathematical concepts from both the theoretical and the empirical perspectives. As the teaching and learning of calculus occurs through the use of language and symbolism, touching on the issues of language in this section is therefore unavoidable.

In his study Monaghan (1991) found that some students' problems related to the understanding of limits stem from the ambiguities inherent in the phrase *'tends to',* and the words *'approaches', 'converges',* and *'limit'*. While the three action verbs are dynamic in the mathematical sense, the word '*limit'* is static. In the mathematical context the stated verbs are associated with limiting processes. However, in the study by Monaghan some students used the word 'approaches' in the static sense, that is, for example, as a way of thinking. In their experiences students had also met the word '*converge*' associated with light rays and now they could not see how in a sequence numbers could converge. For convergence of light rays to occur, all light rays converge at the same time. But the convergence of the terms of a sequence is a result of observing what happens to the terms of the sequence, as $n \to \infty$. There are also some inconsistencies in mathematical contexts concerning the use of the term 'approach' or 'tend to'. For example, in a case of a constant function, the limit is a constant and the process of approaching becomes invisible as the number that is said to be approached is already reached in the sense of landing on it.

Monaghan is not the only researcher who has pointed out the problems of language in learning limits. Other researchers who have associated problems of understanding limits with language include Taback (1975), Tall and Schwarzenberger (1978), and Davis and Vinner (1986). Taback (1975: p. 111) says that "the word 'reach' in the context of limits as used by the mathematicians refers to the neighbourhood of a point but to some pupils it refers to landing on a point". There are some mathematicians, for example, Tall (1991), who interpret the word 'reach' in the sense of landing on a point. In an example related to the generic extension principle, Tall gives a description that a convergent sequence described by the simple formula such as $\frac{1}{n}$, tends to the limit 0, but the terms never equal the limit. That is, the terms never reach the limit value. Tall and Schwarzenberger (1978) suggest that the phrase 'as close as we please' lacks precision in that it does not show by how close one can be in quantitative terms. They further suggest that the interpretation of the word 'close' is also problematic in that it suggests being near but not coincident; if it meant coincident, it could have been stated explicitly.

Cottrill *et al.* (1996) analysed data from 25 interviewees who were the university students from a calculus course. The main focus of the instruction in this class was to assist students to make the mental constructions which they thought were necessary in understanding the limit concept. These constructions commonly known by the acronym APOS, are stated and discussed in chapter 3, the theoretical framework chapter. Computer technology was used to assist the students. The findings reflect that some problems encountered by students were related to conceptions such as:

- The limit of a function at a point means the value of the function at that point which the students may conclude from studying a lot of continuous functions;
- Failure to interiorise an action into a process (to be elaborated in chapter 3). For example, students would consider one or two values in the neighbourhood of a ($x \to a$), and never look at several values approaching a;
- Failure to differentiate between the limit as a static object and the limit process; and
- Failure to co-ordinate the domain process ($x \to a$) with the range process ($f(x) \to L$), to form a coordinated process schema.

The researchers believed that if computers can make these constructions, the students will also make the corresponding constructions in their minds. But it seems that it was not necessarily the case. Giving the computer instructions to perform a task is not the same as when the task is performed by an individual. The effects of the exercise will probably depend on how an individual interprets the results obtained through the use of the computer.

White and Mitchelmore (1996) suggest that students' difficulties in understanding calculus concepts, in particular the limit concept, lie in their undeveloped conception of a variable. Students treat variables as symbols to be manipulated rather than quantities to be related. Frid (2004) confirmed this. Subjects saw the role of symbols as manipulation. Hence, they could not make relations between these symbols and the quantities they represented. Orton (1983b) found that students could not find the relationship between the slopes of secants and slopes of tangents. In a study by Aspinwall and Miller (2001) students' conception of the relationship between the derivative and the slope of the tangent line was confined to formulas only. Conceptions on relations that exist in other forms of representation were lacking.

The understanding of real number, infinity and infinitesimals are also a stumbling block to the understanding of limits of functions (Tall & Schwarzenberger, 1978; Sierpinska, 1987). In limits of functions these concepts are important for the understanding of the domain processes, $x \to 0$ and $x \to \infty$, the range process, $f(x) \to L$, neighbourhood of a point, integration as the limit of a sum, and sequences involving numbers with infinite decimals. Orton (1983a, 1983b) reports that most students treat '∞' as an algebraic symbol which can be manipulated in the same way as the usual letters which stand for numbers in algebra. This view is confirmed by Sierpinska (1987). In her lesson a student argued that 999… divided by 999… is 1. The reason given by the student being that 999… equals infinity, so infinity divided by infinity is 1, from a conception that a number divided by itself is 1. Here the subject confuses the concept of potential infinity with the actual infinity. Infinity as potential is seen in expressions such as $n \to \infty$ or $x \to \infty$.

Infinity as actual can be explained from the theoretical perspective of George Cantor, through the use of ordinal and cardinal numbers. A set of counting numbers {1, 2, 3, 4, …} has an infinite number of elements. This set also has the same number of elements as the set {2, 4, 6, 8, …}. This is because if the elements of the two sets can be put in one-to-one correspondence, each number will have a number that it matches with. Intuitively, since the odd numbers are missing in the second set, one could say that the number of elements in the second set is less than the number of elements in the first set. These two kinds of infinity were also a problem in the past. Mathematicians of the past also experienced the concepts of infinitely large as an epistemological obstacle. As already discussed in the previous section, Aristotle and Kronecker believed that infinity is potential and it is never an actual number (Hollingdale, 1989).

In a study by Davis and Vinner (1986), calculus university students were given a test in which one of the questions required them to give a description of the limit of a sequence in intuitive or informal terms. The findings reflect some of the observed misconceptions in students' work were that:

- A sequence must not reach or attain its limit,
- The interpretation of the phrase 'going towards a limit' to carry everyday meaning which would lead them to perceiving the sequence 1, 1, 1, 1, … as divergent since its terms are not going towards anything,
- Confusing limit with bound, assuming that the limit has the last term, a_∞;
- That one can go through infinitely many terms of a sequence; and
- The assumption that the sequences must have some obvious consistent pattern. Hence, the sequence such as 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 0, … is immediately rejected as a sequence. (ibid., p. 294).

The problem of whether a limit is attainable has not only been the problem of the past. It seems to cause the slow development of the limit concept even in educational practice today. The studies of Juter (2003a, 2003b; 2004; 2005) confirm this. In her 2004 study, she asked the first year university students if the function $f(x) = 2x + 3$ could attain the limit value as $x \to 3$. Some subjects said that this function cannot attain its limit because the definition says it can come only close. But we know that this function is a continuous function and therefore can attain its limit. That is, $\lim_{x\to 3}(2x+3) = f(3)$. The substitution of 3 in the formula does cause confusion because the definition indeed refers to x tending to 3 and not x being 3. Tall (1996) has also pointed out to the fact that this contradiction between substitution of $x=a$ and the informal definition is the source of cognitive conflict.

Everyday meaning of some words which appear in mathematical contexts with unique meanings also cause sluggishness in the knowledge to be acquired. In the past some mathematicians thought that infinity is something that could be achieved with patience.

Hence, it is also not surprising that students believe that one can go through infinitely many points of a sequence. Confusing function values with limit values is also a problem that may have been brought about by the concept of a continuous function. This is because for this type of functions, limit values equal the function values. This however, does not mean that the limit values are indeed the function values. Davis and Vinner (1986) attribute the last misconception to the influence of specific examples. Having interacted with monotonic sequences, subjects may consider every sequence either to increase or decrease. If a different situation is encountered problems may arise.
Davis and Vinner refer to these conceptions as naïve misconceptions. The two points which they raise in connection with the existence of naïve conceptions being that:

- First, students build up knowledge of representation of structures in their own minds, assembling pieces synthesised from previous experience. The teacher can influence this construction process, but the teacher cannot control it, in part because no teacher has adequate control of the students' previous experience; and
- Second and more important, students already possess conceptualisations that conflict with the new ideas that the teacher hopes will be learned. (ibid., p. 283).

Thus in this case existing mental structures are seen as epistemological obstacles. This is because depending on how they are built, they may retard the process of knowledge acquisition.

Davis and Vinner's view is that when students give wrong answers it is not a problem of lack of an idea but a problem of the selection of the appropriate schema:

> The error is retrieval or choice error…. Thus the presentation by a student of an old (and incorrect) idea cannot be taken as the evidence that the student does NOT know the correct idea…. What is at stake is not the possession or non-possession of the new idea; but rather the selection (often unconscious) of which one to retrieve. (ibid., p. 284).

In a study by Williams (1991), students were asked the question requiring them to describe what they understood the limit to be. Students' responses were classified into four kinds of conceptions. The students viewed the limit as dynamic, as unreachable, as a bound, and as an approximation. The interpretation of the limit as unreachable or as a bound is similar to spontaneous conceptions of limit provided by Cornu (1991). This shows that everyday meaning of words with dual meaning can be an epistemological obstacle. However, it is true that the informal definition of limit does carry with it the implied meaning that we approach the limit and never reach it. As pointed earlier, this implied meaning causes confusion to students (Tall, 1996).

The idea of integration also gives students some problems. The idea of integration can be interpreted as a way of finding the area under a graph by limiting processes. Such an area can be visualized and obtained by adding up the approximate areas of the thin strips under the graph, which is taken to be the limit of these sums. In addition to this, Tall (1996, p. 312) points out that in considering the lower sums and the upper sums for the function $y=x^3$ from 0 to 1, by taking more rectangles, some students think that "as long as the rectangles have a thickness, they do not fill up the surface under the curve, and when they become reduced to lines, their area equals to 0s and cannot be added". The error that students make in this particular case is failure to take the width of the rectangles to be tending to zero, as the number of rectangles tend to infinity. Thus the concept 'infinitesimally small' poses problems as its conception is above the sense perception. This has also been a problem of the past.

The chapter has highlighted some of the reasons that led to the slow development of the limit concept in the past. It has gone further to discuss the epistemological obstacles that students encounter in the educational practice today in light of the obstacles of the past. The next chapter discusses the theories that were used in interpreting the collected data for the study.

Theoretical Framework

The previous chapter has reviewed a body of literature in which researchers have investigated epistemological obstacles that mathematics students at undergraduate level encounter in understanding the idea of limit. This chapter presents and discusses the framework within which the data collected in this study were analysed. Two theories are used in the data analysis. The theory that is used in data analysis in answering the first question of the study is given the acronym APOS. The A stands for action, P for process, O for object and S for schema. The first question is about the investigation of epistemological obstacles that mathematics students at undergraduate level encounter in coming to understand the limit concept.

The APOS theory is appropriate in analysing data related to the posed question because the theory is specifically introduced as a language for talking about the nature of learning topics such as the limit concept (Cottrill *et al.,* 1996). This framework however, does not say much on the role that language and symbolism play in concept acquisition. It deals more on the mental constructions that are necessary in learning mathematical topics. Because of this, an alternative framework that would take care of data related to the second question on the role of language and symbolism in coming to understand the limit concept was sought. This framework is concerned with the theories on semiotics. This branch of knowledge deals specifically with language and symbolism in learning as reflected in the second part of the chapter. The first section discusses the APOS theory and the second section discusses the theories on semiotics.

APOS Theory

This section gives definitions and descriptions of the three pieces of knowledge: actions, processes and objects. The idea of schema is also discussed as it is the conceptual structure. That is, the way the concepts or pieces of knowledge are related to one another. Since the descriptions are from different theorists, meanings of the terms will be compared and contrasted where need arises. The descriptions are accompanied by examples.

Actions

An action is described as a repeatable mental or physical manipulation of objects. Such a conception would involve, for example, the ability to plug in numbers into an algebraic expression and calculate. It is a static conception in that the subject will tend to think about it one step at a time, that is, think about the single evaluation of an expression at a time (Dubinsky & Harel, 1992). Given the question evaluate $\lim_{x \to \infty} \frac{1}{x}$, and if say some numbers such as 100, 100 000, 1000 000, are substituted for x, the corresponding $f(x)$ values will be 0.01, 0.000 01, and 0.000 001. Now if in finding the limit value the conclusion made is that the limit is zero because these numbers are approximations of zero, such a learner has an action conception. This is because such a conclusion is arrived at by a consideration of a finite number of steps, one at a time.

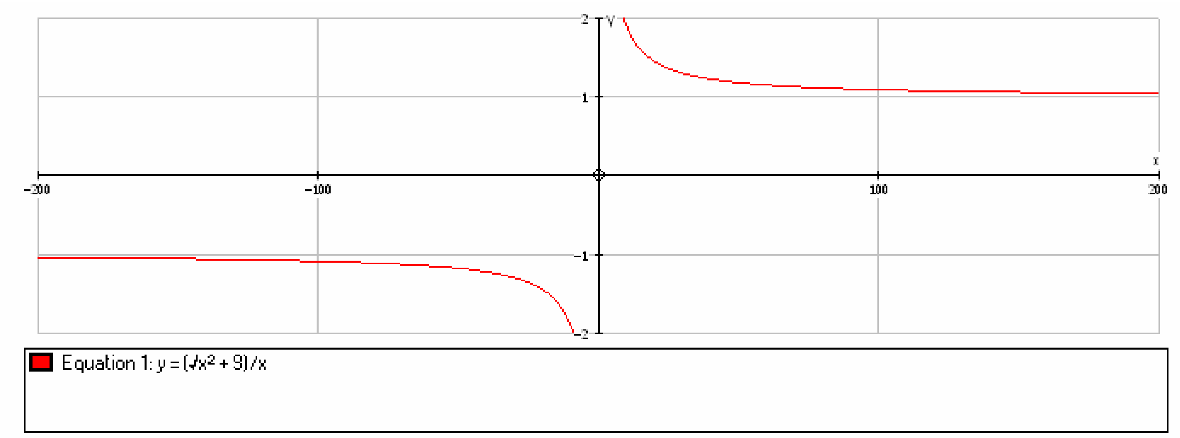
Processes

The calculation of the limit concept involves an infinite number of computations: "…. once a calculation involves an infinite number of steps, it can only be understood through a *process* conception." (Cotrill *et al.,* 1996, p. 173). In evaluating $\lim_{x \to \infty} \frac{1}{x}$, for example, the limit value zero will be obtained by a consideration of an infinite number of computations. This is because not all computations are performed but some are contemplated through the process of 'tending to' (Cotrill *et al.,* 1996).

A process can also be described as a dynamic transformation of quantities according to some repeatable means that, given the same original quantity, will always produce the same transformed quantity (Sfard, 1991). If say we are to find the derivative of the function, $f(x) = \frac{1}{x}$, if we treat it as a quotient, the quotient rule will be applied. If we look at is as a power of x, written as x^{-1}, the power rule will be applied and the same result $f'(x) = -\frac{1}{x^2}$ will be obtained.

Objects

Seeing a mathematical entity as an object means being capable of referring to it as if it was a real thing. It also means being able to recognise the idea "at a glance" and to manipulate it as a whole, without going into details (Sfard, 1991, p. 4). Given the graph of the function satisfying the equation, $f(x)=\frac{\sqrt{x^2+9}}{x}$ (see diagram), one should be in a position to see 'at a glance' that the function tends to 1 as x values tend to positive infinity ($+\infty$), and the function tends to -1 as x tends to negative infinity ($-\infty$), without a consideration of one point at a time.



Typically objects are described by their properties, their relationships with other objects and ways in which they can be used. "We might ascertain whether an individual has constructed a mental object in relation to a concept by the way that individual talks about or writes about the concept" (Tall *et al*., 2000, p. 230). This idea is elaborated more in subsection 3.1.5.

Schemas

A schema is described as a coherent collection of actions, processes, and objects (Cottrill *et al.,* 1996, p.172). Since it is a conceptual structure, actions can be performed on it to form a schema at a higher level. Thus besides the encapsulation of processes, objects can also be formed from schemas. An example of a schema within the limit concept could be the chain rule schema. Within the chain rule schema we also find the function schema and the derivative schema. In constructing the chain rule schema certain mental constructions are necessary. Part of the function schema should have:

- A process and an object conception of a function; and
- A process and an object conception of a composition of a function;

Part of the derivative schema consists of:

- A process conception of differentiation;

- Coordination of the constructed schemas of function, composition of functions, and differentiation to define the chain rule. The coordination consists of first recognising a given function as the composition of two functions and taking their derivatives separately and multiplying them; and
- Application of chain rule to different situations. (Clark *et al*., 1997, pp. 349-350).

This shows how the processes and objects should be connected in order for one to be in a position to apply the chain rule. Other schemas within the limit concept that will be needed in tackling certain mathematical tasks will be presented in chapters 5 and 6 to aid the discussion concerning the data analysis.

Formation of mathematical concepts

Sfard (1991) has suggested two ways in which mathematical concepts can be perceived, structurally as objects and operationally as processes. Sfard suggests that in concept formation for most people the operational conceptions precede the structural. This is said to be true whether in the historical development or in individual learning. For example, the object 'infinity' comes from the process of becoming big, 'infinitesimal' comes from the process of becoming small and 'limit value' comes from the process of 'approaching' or 'tending to'. Sfard sees structural conceptions as static, instantaneous and integrative whereas the operational conceptions are dynamic, sequential and detailed.

This process-object nature of mathematical concepts is also applicable in algebra. An algebraic representation can be interpreted both ways, structurally and operationally. It may be explained operationally as a concise description of a computation or structurally as a static relation between two magnitudes. This corresponds with the dual role of the equal sign "=". It can be seen as a symbol of identity. It can also be seen as a command for executing operations (Sfard, 1991). The example in the next paragraph illustrates this.

The symbolism, $\lim_{n\to\infty} a_n$, can represent both the process of tending to the limit and the limit value (Tall, 1991b). In case where a_n is equal to $\frac{1}{n}$ we can have $\lim_{n\to\infty}\frac{1}{n} = 0$, since the two represent the same mathematical object and the equality sign relates the two sides. Thus we may also talk about $\lim_{n\to\infty}\frac{1}{n}$ being smaller than $\lim_{n\to\infty}\frac{n}{n+1}$ to be a true statement since $\lim_{n\to\infty}\frac{1}{n}$ is 0 and $\lim_{n\to\infty}\frac{n}{n+1}$ is 1 and 0 is less than 1.

Dubinsky (1991) suggests five ways in which processes and objects are constructed mentally from existing ones. These are interiorisation, coordination, encapsulation, generalisation and reversal. Sfard (1991) identifies three, namely, interiorisation, condensation and reification. The five ways of Dubinsky are discussed first. These are followed by a discussion of the three ways mentioned by Sfard.

- Interiorisation

When an individual reflects upon an action, he or she may begin to establish conscious control over it. The action would then be interiorised as it becomes a process. The action of evaluating the range values from the domain values via the function may be interiorised into processes if not all computations of the limiting process are performed but some are contemplated. If fact, since the computations are not finite, the limit value can only be obtained through the interiorisation of actions into processes.

As shown earlier, in finding the limit of say the function $f(x)=\frac{1}{x}$ *as* $x\to\infty$, a table of values may be generated, but as this is done, it is of course not possible to compute all the function values as the domain itself consists of an infinite number of values. A conclusion about the limit value will therefore be arrived at by using statements such as '$f(x)$ tends to…as x tends to', statements which show a consciousness that not all computations are actually performed. This is because the process of 'tending to' goes on and on and never stops.

- Coordination

This is the construction of a process by the coordination of two or more other processes. Two functions could be coordinated to form a composite function. The coordinated process, $f(x) \to L$ *as* $x \to a$, or $a_n \to L$ *as* $n \to \infty$, is achieved through the coordination of the domain process, $x \to a$, and the range process, $f(x) \to L$, via the function (Cottrill *et al.,* 1996).

- Encapsulation

It is the construction of the object through a process. This is achieved when an individual is aware of the totality of the process, realises that transformations can act on it and is able to construct such transformations. We can talk about the limiting processes such as 'tending to' or 'approaching' being encapsulated into limit values or limit points. We can also talk about adding the limit values of different functions or subtracting one limit value from the other, descriptions that show that we are now treating these entities as fully fleshed objects or nouns.

- Generalisation

This is the ability to apply an existing schema on a greater range of phenomena. It is a passage from one to many (Dubinsky, 1991). If say we have to find the derivative of the function, $f(x) = (2x+3)^2$, a composite function, a chain rule has to be applied. If the second example of a composite function is given and one has to find the derivative and the chain rule is applied, an application of the chain rule schema is being extended. If more examples of composite functions are given and now one is in a position to realise that the chain rule schema can be applied to all composite functions, a generalisation on the application of the schema is made. If however a schema is applied to a situation in which it is not applicable, for example, in differentiating the function, $f(x) = x \ln x$, we say the schema is over generalised. The product rule could be used in differentiating both the first and the latter function. This is because the first function can be written as a product, $f(x) = (2x+3)(2x+3)$. The chain rule is however inapplicable in differentiating the latter function. This is because the latter function is not a composite function.

- Reversal

This is when an individual is able to think of an existing internal process in reverse to construct a new process. The pairs of reverse processes would include: addition and subtraction, multiplication and division, differentiation and integration, etc. Thus in solving equations, for example, one is able to think of processes in reverse. Where there is subtraction, addition will be employed and vice-versa. Where there is multiplication, division will be employed and vice versa, etc.

Closely related to the work of Dubinsky (1991) with regard to the way mathematical concepts are formed is the work of Sfard (1991). She has identified three stages that are necessary in concept formation. These are:

- Interiorisation

When a process has been interiorised the individual can carry it out through mental representations without actually performing it. This is a similar description to that of Dubinsky (1991). Hence, the given examples in discussing the work of Dubinsky also hold in the case of Sfard.

- Condensation

Ability to deal with a given process in terms of input-output without necessarily considering its components; thus finding the limit of the functions $f(x) = 2x + 8$ and $f(x) = 2(x + 4)$ as x tends to a, for example, are the same process (Sfard, 1991). This is because in computing their limits, the same limit values will be obtained without necessarily following the same sequence of intermediate procedures or steps. In finding the limit of $f(x) = 3x + 2$ as x tends to 2 which is 8, we will also take this as a process by considering only the first stage of the function and the limit value without considering the intermediate steps that gave rise to 8, the output.

- Reification

This is the conversion or translation of a condensed process into an object. That is, transition from an operational to a structural phase of concept development. A process is reified into an object if an object is detached from the process that gave rise to it. This is similar to encapsulation of a process by Dubinsky (1991) and Cottrill *et al.* (1996).

In constructing actions, processes, and objects to form a limit concept schema, Cotrill *et al*. (1996) list seven mental constructions that may occur:

1. The action of evaluating f at a single point x that is considered to be close to or even equal to a;
2. The action of evaluating the function f at a few points, each successive point closer to a than was the previous point;
3. Construction of coordinated schema as follows:
   - (a) Interiorisation of step 2 to construct a domain process in which x approaches a.
   - (b) Construction of a range process in which y approaches L.
   - (c) Coordination of (a) and (b) via f. That is the function f is applied to the process of x approaching a, to obtain the process $f(x)$ approaching L;
4. Perform actions on the schema by talking about the limit as a fully fleshed object or a noun. In this way the schema of step 3 will be encapsulated to become an object.
5. Reconstruct the processes of step 3(c) in terms of intervals and inequalities. This is done by introducing numerical estimates of the closeness of approach, in symbols, $0<|x-a|<\delta$ and $|f(x)-L|<\varepsilon$;
6. Apply a quantification schema to connect the reconstructed process of the previous step to obtain the formal definition of a limit; and
7. A completed $\varepsilon-\delta$ conception applied to specific situation (pp. 177-178).

The last three stages involving ε-δ definition are excluded in this study. This is because these stages constitute part of the content that was not covered by the subjects in this study. In order to see how the researcher has interpreted the stages to be considered in this study (stages 1 to 4) one has to put them into context. Suppose the question asked is:

Find $\lim_{x \to 2} \frac{1}{x}$

Stage 1

In applying stage 1 to the given question, one has to be in a position to choose a value that is close or in the neighbourhood of 2. Substitute it in the expression representing the function. Such a value could be taken as 2.001. When substituting 2.001 for x we get $\frac{1}{2.001} = 0.4999$. Similarly when substituting $x = 2$, since $a = 2$, we get $\frac{1}{2} = 0.5$.

Stage 2

Here we consider the points in the neighbourhood of 2, whether from the left hand side or from the right hand side. The choice should be made in such a way that each time we make a choice we get closer and closer to 2. Such numbers could be: 2.01, 2.001, 2.0001, … when 2 is approached from the right. We could also have: 1.9, 1.99, 1.999… when 2 is approached from the left. Substituting this numbers for x will yield:

x	1.9	1.99	1.999 →	…	**2**	← …	2.0001	2.001	2.01
$\frac{1}{x}$	0.5263	0.5025	0.5002 →			←	0.4999	0.4997	0.4975

Stage 3 (a)

One has to realise that the process of approaching 2 cannot be done by considering a finite number of x's approaching 2, but by considering an infinite number of them. This is because the number of such x values is infinitely many. Each time we choose an x value close to 2, there is another one that is closer to 2 than the previous one. Thus the domain process will be formed.

Stage 3(b)

Watch what happens to the *y* values. These values now approach 0.5 from both the left and the right. This is called the range process. It is constituted by the images of the domain process.

Stage 3 (c)

We now form a coordinated pair of processes (domain process and the range process) by making an observation that the process of *x* values tending to 2, occur simultaneously with the process of *y* tending to 0.5, the limit value.

Stage 4

Since the limit value has now been obtained. We can now talk about it without making reference to how it was obtained. In this case we can say 0.5, the limit value, is a rational number. We can also say that it is less than 2. In all these cases we refer to the limit value as a real object or a noun.

Within this framework an analysis of emergence of epistemological obstacle in coming to understand the limit concept will follow the order of concept formation mentioned by Cottrill *et al.* and Dubinsky. The three stages are:

- Interiorisation of actions into a processes;
- Construction of coordinated processes; and
- Encapsulation of processes into objects.

Having discussed the theory that suits the nature of the limit concept, the next section looks at how the semioticians perceive the role of language and symbolism in learning concepts. In this case the limit concept is the object of discussion.

Language and symbolism in Mathematics: A Semiotic perspective

Chapter two discussed the role of language and symbolism in understanding the limit concept from both the theoretical and the empirical perspectives. Epistemological obstacles encountered in learning the limit concept were also discussed. This section discusses language issues in learning mathematical concepts from the semiotic theoretical perspective. The discussion in this chapter concentrates on the theories of Chapman, Winsløw, and Steinbring. This is because the work of the stated authors approaches the role of language and symbolism from the semiotic perspective. Hence, their theories complement each other. As the previous section has discussed a theory which is more inclined to cognitive aspects of knowledge acquisition, this section discusses a theory that includes both the cognitive and the social aspects of knowledge acquisition.

Semiotics is the study or the science of signs (Chapman, 1993; Winsløw, 2000), and mathematical concepts rely on the intensive use of signs. A sign is something that stands for something else called the reference object (Chapman, 1993; Steinbring, 2002). Language and symbols are signs since they stand for objects or reference contexts that they represent (Steinbring, 2002). Thus Steinbring uses the term sign and symbol interchangeably. For the purpose of this study the word symbol will be used.

According to Steinbring (2002), a characterisation of the role of mathematical symbol requires a consideration of two functions, a semiotic function and an epistemological function. A semiotic function is the role of the mathematical symbol to stand for something else and the epistemological function is a consideration of the mathematical sign in the frame of epistemological interpretation of mathematical knowledge. The semiotic function is represented by the asymmetric relation between a symbol and an object or reference context as:

Figure 3.1 The object-symbol relation

Connecting this diagram to one of the representations or symbol used for the function in section 3.1.3 we would yield the following diagram:

Figure 3.2 The function-symbol relation

“A mathematical object, such as a function, does not exist independently of the totality of its possible representations, but it is not to be confused with any particular representation either” (ibid., 2002, p. 3). Thus the algebraic symbol used for the function is not the function itself but it is a representation of a function. There are other forms of representation of a function that exist, the graphical, the tabular, and the verbal modes of representation. In order to form the function concept there has to be some activities that have to be introduced to relate the function to its representations as will be discussed or shown in the subsequent discussions.

Steinbring shows that symbols have no meaning of their own. In order to bring about meaning or concept formation, an epistemological subject has to be introduced by the establishment of mediation to suitable reference context, e.g., suitable mathematical activities could be introduced within the specified reference context. Steinbring suggests that the connection between the reference context, the symbol, and the mediation can be represented by an epistemological triangle as follows:

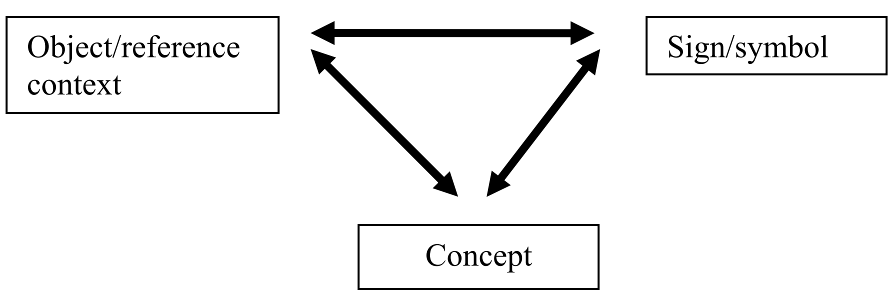
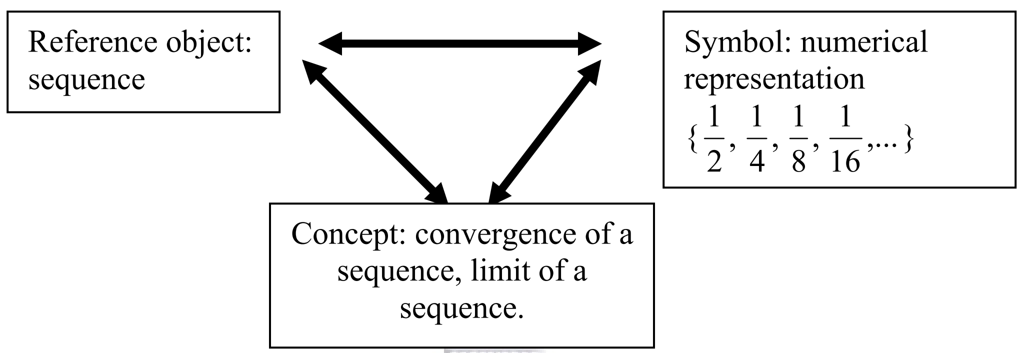


Figure 3.3 An epistemological triangle

The double arrows show that during the process of concept formation there is a movement in all directions. That is, there has to be connections made between all components of the triangle.

To demonstrate how the epistemological triangle operates reference is made to the example of a converging sequence used in chapter two. Epistemological issues here concern making connections between the symbol and the reference object.



In finding relations between the symbolism used and the reference object, sequence, the same questions as those posed in promoting understanding could be used as some form of mediation. This is because understanding is also described as making connections. The questions are:

1. What happens to the terms of the given sequence as $n \to \infty$?
2. Does the given sequence converge or diverge?
3. What is the limit of the given sequence?
4. Can the given sequence attain its limit value?

In responding to the stated questions, one will have to refer to the given representation of the sequence and make the following observations:

- That the terms of the sequence approach 0;
- That the sequence converges;

- That the limit of the sequence is 0, and
- That the given sequence cannot attain its limit. The terms will keep on approaching 0, but they never reach it.

Winsløw (2000) refers to the mathematical symbol as the signifier, and the object is referred to as the signified. Winsløw puts emphasis on the necessity for using more than one representation as a condition for promoting understanding in concept formation. Different modes of representation of the same object preserve the object they represent, that is, the object they represent does not change even though the representations are different. Winsløw points out that even when the pertinent registers of representation are available the possibility of changing between these modes of representation remains an important operation for both the conception and handling of mathematical objects. This is because each representation has its own semantic qualities and relations. If only one mode of representation is used the schema will be too restricted as its applicability will be confined to one mode of representation. But if more representations are used a more functional schema will be formed.

In the context of limit, some of the Object-Preserving Transformations (OPT's) include translating from one form of representation to another, rationalisation, factoring and other forms of simplification of expressions. Translating from one representation to another is a transformation that preserves the reference to the common object. As shown earlier the algebraic and the graphical representations have signified the same function. Janvier (1987) gives a list of some translation processes as: interpretation, computing, sketching, and parameter recognition. In rationalising either the numerator or the denominator in an expression, we multiply the expression by a special form of one. Multiplying by one leaves the object unchanged or preserved. Factoring just rearranges the surface structure to equivalent forms. Thus the semantic structure of the represented object remains unchanged.

As Chapman (1993) suggests learning mathematics involves learning its register. She describes a register as a particular kind of language used in a specific situational context.

According to Chapman a register has both the thematic context and the interactional context. From the extract that follows, Chapman identifies both the theme and the type of interaction that takes place in the discussion:

Teacher:	OK. Work out through it step by step. Remember to label both axes.
Daniel:	Should we put in all the points?
Teacher:	Of course we put in all the points. (ibid.).

The thematic aspects of the discussion here are drawing graphs. This task is aided by the discussion or the interaction between the teacher and the learner. Where Daniel is not clear as to what should be done, he asks a question for clarification from the teacher. The teacher responds to the question in such a way that it gives Daniel some direction to take.

Having discussed the theories that will guide the data analysis, the next chapter discusses the methods that were used in implementing the study.

Research Design and Methodology

The previous chapter has discussed the theories used in analysing and interpreting the data for the study. This chapter discusses the methodology that was employed in implementing the actual study. Since the methodology for this study has resulted from the lessons learned from conducting the pilot study, first the experiences of the piloting are presented.

The pilot study

The subjects that served as the sample in piloting are those who were admitted into FOST in the academic year 2002/2003. In the first year of study there were about 200 subjects. The subjects were followed up to their second year of study in which a class of about 50 subjects took part in the research project. The purpose of conducting the pilot study was to:

- Check the suitability of the chosen research design,
- Check the suitability of the research instruments;
- Check the suitability of the research questions;
- Check the suitability of content for the study and the subjects;
- Check if the questions set displayed the behaviour that could be explained with the language used in the theoretical framework;
- Develop the proper skills of interviewing; and
- Check if the theoretical framework chosen was in alignment with the research questions and methods of data collection.

A case study design employing questionnaires, interviews, and non-participant observation was used for the pilot phase of the study. The theoretical framework used was based on the theories of knowledge from the work of philosophers Locke, Kant and Plato. Some lessons learned from the pilot study phase led to effecting some changes in the study phase. The discussions that follow focus on the changes made in methods of data collection, theoretical framework, reliability and validity issues, and data analysis. But first the question of how the case study was chosen as the appropriate research design and some lessons learned from the pilot phase are discussed.

The case study design

The choice of the research design for the study was based on the nature of the research questions and the nature of the phenomenon under study. During the pilot study the research questions were phrased as follows:

1. What epistemological obstacles do mathematics students at undergraduate level acquire or overcome in understanding the limit concept?
2. What is the role of language and symbolism in understanding the limit concept?

The stated questions are empirical in nature. Empirical questions are questions that require data to be collected from the real world (Lecompte & Preissle, 1993; Babbie & Mouton 2001). In order to respond to the stated questions mathematics students at undergraduate level are the primary source of data. They have to be asked questions from which their responses will reflect how they acquire or overcome epistemological obstacles in understanding the limit concept. They also have to be asked questions which will require their responses to reflect how they use language and symbolism in understanding the limit concept. These questions would be non-empirical if they were phrased as follows: What is an epistemological obstacle? What is language? or What is symbolism? The latter questions are non-empirical because in order to answer them one has to examine and analyse the body of scientific knowledge (Babbie & Mouton, 2001).

Since the stated questions are empirical, it means that an empirical research design had to be chosen. But there are quite a number of empirical research designs. Among the empirical designs that exist, the following could be mentioned: surveys, experiments, and case studies (Lecompte & Preissle, 1993; Babbie & Mouton, 2001). So, what differentiates a case study from these other research designs? A case study is the examination of a specific phenomenon such as a program, an event, a person, a group, an institution or a social group (Yin, 1989; Merriam, 1988; Lecompte & Preissle, 1993). By contrast survey analysis usually addresses fewer individual aspects of phenomena, but does so across far more instances (Lecompte & Preissle, 1993). A case study researcher has no control over the variables of interest while in experimental design the variables of interest can be manipulated (Merriam, 1988).

In this study the phenomenon under investigation was epistemological obstacles that mathematics students at undergraduate level encounter in understanding the limit concept. How this group uses language and symbolism in their understanding of the limit concept also formed part of the investigation. The main focus of the study was mathematics students at undergraduate level because the literature has confirmed that students at this level have problems in understanding fundamental calculus concepts, in particular the limit concept. The mathematics students at undergraduate level at NUL also had the problem of high failure rate in calculus. Because of the reasons discussed above, the case study design therefore was the most appropriate research design for the study. Since the nature of the research questions in the actual study is the same as those in the pilot stage, the research design was suitable for both stages of research.

The next subsection discusses the instruments that were used, how they were constructed and their efficacy in revealing the desirable behaviour under study.

Research instruments

Unlike experimental or survey research designs, case study does not claim any particular methods of data collection (Merriam, 1988). The methods of data collection in this study were simply chosen on the basis of the type of data that were appropriate for the investigation of the phenomenon under study. The appropriate instruments in this case were those that could allow students to write and talk. This is because one could be in a position to infer from their responses the epistemological obstacles that they encounter and also the role of language and symbolism would be visible from their work. To allow an opportunity for writing questionnaires were used. To allow the opportunity for talk, interviews were used. Subjects were also given tasks which required them to talk and write while the researcher acted as a non-participant observer.

Questionnaires

Two questionnaires were constructed. The first questionnaire covered questions on the limit of a function. The concepts covered in the questionnaire were on derivatives, integrals, limits, and continuity. The second questionnaire covered the limit of a sequence. Only the numerical and the algebraic modes of representation of the sequence were used. Some questions asked were open and as a result it became very difficult to develop the categories of responses as they were too varied. This experience led to providing students with options from which they had to choose appropriate answers. From Questionnaire 1, Question 5 was among the questions that were open ended. The question is now presented:

Question 5
Use the given table to answer the questions that follow

x	$f(x)$
0.1	1.1
0.01	1.01
0.001	1.001
0.000 1	1.000 1
0.000 01	1.000 01
…	…
-0.000 01	0.9999999
-0.000 1	0.999999
0.001	0.9999
0.01	0.999

(a) What is the limit of the function $f(x)$ as $x \to 0$? How did you get your answer?
(b) What does it mean to say that the value you have found is the limit of the given function $f(x)$ as $x \to 0$?

In responding to question (a) the explanations given were too diverse. Hence, it was difficult to develop the categories for the responses. Because of this, a question that still tested the subjects' conception of limit was constructed. But this time six options were provided in order to ease the process of categorisation. Question 5 was therefore replaced by Question 4 in questionnaire 1 of the actual study. The question was restructured as follows:

Question 4: How can we see if a function $y = f(x)$ has a limit L as x is approaching 0?

It is by:

1. **Calculating y for $x = 0$, i.e. calculate $f(0)$**
2. **Calculating $f(1)$, $f(2)$, $f(3)$ and so on and observe the results**
3. **Calculating $f(x)$ for $x = 1/2, 1/4, 1/8$ and so on**
4. **Substituting x by 0 in the function formula, and calculate the value.**
5. **Substituting numbers that are very close to 0 for x in the formula and look for the value of y.**
6. **Substituting numbers that are very close to 0 for x in the formula and look for the value of y that is being approached as x values approach 0.**

(*Choose the option(s) that best describes your answer*).

Why will you do so?

The advantage that this question has over the former is that it has options from which the subjects can choose their answers. This makes the analysis more focused to a few options rather than many.

Interviews

Interview questions emerged from the subjects' questionnaire responses. During interviews very early in the discussions, the subjects were asked questions that helped them to overcome epistemological obstacles. This led to difficulties in investigating conceptions related to errors that were displayed in the responses to the questions asked at a later stage, since conceptions that gave rise to them were already appropriated.

In order to overcome this problem, in the actual study the interview questions were asked in such a way that students were not forced into a situation of overcoming epistemological obstacles before questions related to the existence of the displayed misconceptions were asked and answered. It must be acknowledged that this was a very difficult task. Hence, even in the main study there were some cases where such instances, though not very many, still occurred.

Non-Participant observation

The subjects were given the tasks in which their discussions were audio taped and the researcher acted as a non-participant observer. The purpose of the task was to find out how the subjects acquire epistemological obstacles and also how they overcome them without the intervention of the teacher. A difficulty that arose here was that in some cases during the discussions there were issues that could be clarified more through probing or using intervention of some kind, but since the researcher was not supposed to participate, it became very difficult to trace the sources of some of the difficulties or errors. For example, in a group interview, three

subjects were given a task which required them to find that the tangent line is the limit of the secant lines. The question appears on the next page.

An extract showing a situation where the researcher felt some need to intervene but could not do so as a non-participant observer now follows:

S179: **The question is 'how many secants can you add to the diagram?'**

S203: **Will that mean that we are going to count only those that we have added to the diagram?**

S179: **No we can add more but now we have added only 3.**

S203: **How many secants have we added to the circle? They can occupy each part of the circle except when they touch the circle now they become tangents. So by the time they become parallel or are the same line with the tangent they are no longer secants. So I think there are many secants that we can add.**

S167: **I think I am not so sure about the number.**

S203: **But the tangent has to be excluded.**

S179: **Yes we have to exclude the tangent.**

S203: **It is ten to the power …**

S179: **(Interrupting) There are as many of them. Infinite I think.**

S203: **We can have as many as the points on the circle.**

S167: **From my point of view I think we have to specify the number.**

S179: **Yes because the question says "how many secants can you add?"**

S167: **We can even have another one here (pointing at a space without a secant).**

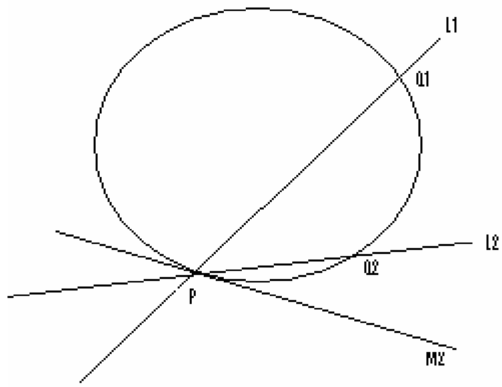
S203: **I think the number of secants we can add will depend on the sharpness of the pencil. But another thing is we shall have secants as the number of points on the circle. But the sharpness of the pencil will determine their number. It can be as many points on the circumference except for the one touched by the tangent. What if we write the answer this way?**

S167: **… But it is somehow ridiculous to talk about the point when we do not know its magnitude.**

The Tangent Problem
Group Task

Instructions:

- **Read the question carefully; and**
- **Explain each step that you take through talking to your group mates.**



Line segments PL1 and PL2 are secants. Secants cross the circle at two points. Line segments PQ1 and PQ2 are chords. Line segment PM2 is the tangent. The tangent touches the circle at one point.
Draw a diagram like the one shown:

1. **Add more secants to your diagram. How many secants can you add to the diagram?**
2. **What happens to the lengths of the chords as the secants get closer and closer to the tangent?**
3. **How far away from the tangent can the nearest secant be?**
4. **Which steepness would you take to be the steepness of the circle at point P? Is it the steepness of the secants or the steepness of the tangent? Support your answer.**

It was interesting to realise that when S179 said that there are infinite number of secants that could be added, S167 said that they had to specify the number. Meaning that infinite in this case was not taken as an actual number. At this point the researcher felt the need to find out about the subjects' conception of infinite, but since the method of data collection was non-participant observer, questions could not be asked. The idea of determining the number of lines by the sharpness of the pencil was also interesting. S167 brought about the idea of the magnitude of a point not being known. A follow up to this conception might have brought up a very interesting discussion because of the contradictions this view has in mathematics. A point is said to have no size, yet a line is made up of infinite number of points. How can something that has no size produce something that has a size? These are some of the ideas of interest that could have been explored further. This situation was

improved by conducting group interviews in which the discussion was driven not only by the questionnaire but also by the researcher.

Research questions

The first research questions focused on the moments of either acquiring or overcoming epistemological obstacles by the subjects. While moments of overcoming epistemological obstacles could be encouraged by the nature of the discussions held, the moments of the acquisition of obstacles were very difficult to encounter. When students were asked situations in which they came to know certain ideas, in some cases such moments were sometimes not remembered. As already explained in chapter 2, knowledge acquisition occurs in a very complex system of interaction. Because of this, it is sometimes difficult to know which part of the system has contributed to a certain kind of knowing. Concepts are also acquired over time and it is difficult to account for the processes that gave rise to them. So, the part which refers to the acquisition or overcoming of the epistemological obstacles was changed to 'epistemological obstacles that students encounter' in coming to understand the limit concept. The focus was no longer on when they were acquired but on their existence or emergence. The change was also effected on the last part on understanding the limit concept to investigating epistemological obstacles in coming to understand the limit concept. This change was brought about by the nature of the theoretical framework used. The analysis is done based on the stages of development of the limit concept. Hence, using the phrase 'in coming to understand' seemed appropriate. The question on the role of language and symbolism was not changed as it did not seem to have any problems.

Content covered

Within the context of a function, the content covered was limit, continuity, derivative, and integral. Though the stated concepts can be expressed or explained in terms of the limit concept, they were simply too many to allow for depth and focus. At one point in the analysis one would concentrate on the concept of derivative, at another on integral, and on the concept of continuity. This made the discussions in data analysis very isolated. Hence, a decision to treat the idea of limit as a concept in its own right was made.

Data collection

The data for the first questionnaire was collected in April 2003 of the academic year 2002/2003. This was during the second semester of the first year. During the first semester students do a pre-calculus course and the calculus course is done during the second semester. A group of about 200 subjects responded to the questionnaire. Fifteen subjects were interviewed individually and in groups within a period of two weeks from the day in which the questionnaire was administered. The interview questions emerged from the subjects' questionnaire responses. Questionnaire 2 was administered in October in the academic year 2003/2004 to a class of about 50 students. This is the time in the first semester of the university academic year. Fifteen subjects also in this case were interviewed individually and in groups. The first questionnaire was administered in the morning hours while the second questionnaire was administered in the afternoon. In responding to the second questionnaire students had just written a test and the researcher felt that it was not given the attention it deserved. This situation was encouraged to happen by a very tight timetable for the science students. In the actual study the questionnaire was administered on Saturday in the morning hours. Permission to administer the questionnaire on this day was sought from the subjects themselves.

Theoretical framework

The theoretical framework used was based on the theories of knowledge from the work of philosophers Locke, Kant and Plato. These theories concentrated more on differentiating between pure (a priori) and empirical (a posteriori) knowledge. Locke's view is that intuitive knowledge is superior to knowledge acquired through logic and reason. He describes intuitive knowledge as knowledge that leaves no doubt for hesitation, doubt or examination (Locke, 1689). Having reviewed the literature and found that one of the epistemological obstacles encountered in learning is deceptive intuitive experiences and the knowledge that logic and proof are sources of conviction in knowledge acquisition, it was difficult to implement Locke's view in data analysis. For example, in dealing with sequences there is a question which required subjects to say whether or not 0.999… was less than or equal to one. Most subjects' response was that 0.999… is less than one. There is no

way that one could have supported this to be a superior knowledge achieved with higher degree of certainty. This question has already been discussed and the reason for saying that 0.999… is equal to one has been arrived at logically. That is, the logical deduction has been according to the researcher the most convincing method for the acceptance of the equality. An alternative framework that was in alignment with the nature of the construction of pieces of knowledge constituting the limit concept was sought.

In improving the credibility of the results of the study, reliability and validity issues were also taken into consideration.

Reliability and validity

The focus was on the reliability of the research instruments. This situation required the researcher to award scores to students and to use the scores as a measure of the extent to which the subjects possessed epistemological obstacles. The higher scores were associated with existence of few epistemological obstacles and the lower scores with existence of more epistemological obstacles.

It is apparent that this method did not strengthen the need to reveal the actual students' conceptions in qualitative terms as one was focused on the interpretation of numbers rather than on looking at the actual obstacles that the subjects had in qualitative terms. The use of low inference descriptors (Seale, 1999) was implemented for this reason. This allowed the subjects' conceptions about the limit concept to be explainable in qualitative terms since the verbatim accounts of what subjects said are included.

The preceding subsections included discussions of some reasons that led to abandoning some practices in implementing the actual study. The next section discusses the steps that were taken in implementing the actual study.

The actual study

In this section the population and sample, research instruments, methods of data collection and data analysis, reliability and validity of research findings, ethical issues and limitations of the study are described.

Population and the sample

The population of the study is mathematics students at undergraduate level. The sample used was a group of students admitted in FOST. This is because this group forms part of the population of the study and also that it was an accessible part of the population. As already mentioned the students that served as a sample during the pilot stage were those admitted in the academic year 2002/2003. The actual study was conducted with the students admitted in 2003/2004. Both groups were followed from their first year to their second year of study provided students in these groups took mathematics as their major subject in the second year of study.

Year 1 and year 2 mathematics students were suited to the intent of the study for a variety of reasons. These include:

- **A high failure rate in mathematics, in particular calculus, is experienced mostly at this stage;**

- **The idea of limit is covered at this stage. The limit of a function is taught in the first year of study and the limit of a sequence in the second year of study;**
- **The sample was an accessible group of the population; and**
- **It was also practicable to work with a smaller group to allow depth of coverage for the investigation.**

The first year of the study combines students who are registered for different programmes in the FOST. The majority of students in this group are enrolled for the Bachelor in Science (BSc). Most of the BSc students are direct entrants from the Lesotho high schools system. Most of these students are 18 years old. The other programmes which have the minority of students in this group are the Bachelor of Science Education (BSc. Ed.). This is a group that is mostly constituted by trained secondary school teachers with at least two years of teaching experience. The other programmes are the Bachelor of Agriculture (BSc. Agric), and the Bachelor of Health (BSc. Health). The Agriculture students are also mainly direct entrants from the Lesotho high schools. BSc Health is constituted mainly by students who have been trained as nurses at the diploma level. All these students constitute what

is called the common first year. In this year of study students in different programmes take the same courses. The subjects in the second year are those who will have passed both the calculus and the pre-calculus courses in first year and will have opted for mathematics as their major subject of study.

Having looked at how the sample of the study was chosen and also the type of subjects that constitutes it, the next section discusses the instruments that were used in data collection. The discussion also includes the rationale for making the choice of instruments.

Research instruments

Since non-participant observation as a method of data collection was abandoned, the research instruments that were used in the actual study are questionnaires and interviews. This is also because data in the form of text had to be collected. Learners use words and/or symbols in responding to questions that test the conceptions they hold on specific phenomena. Conceptions cannot be quantified. Quantification can only be done to the qualitative textual data that represents the conceptions. The choice of questionnaires and interviews were therefore appropriate.

For this study one questionnaire was constructed. All questions were constructed in such a way that they encouraged the evocation of mental constructions that fitted the APOS framework. This framework deals with the way pieces of knowledge that constitute the limit concept are constructed.

The questionnaire also accommodated questions that would reflect the role of language and symbolism in understanding the limit concept from the semiotic perspective. Since questionnaires have a disadvantage in that they cannot probe deeply into respondents' proper conceptions; once the questions are set they remain as they are, interview questions were constructed for this reason. All interview questions emerged from the questionnaire responses and were used in investigating deeper the conceptions related to the errors committed by the subjects. The probing of responses in the interviews was dependent on the type of answer that the subject gave during the interview. In interviewing the subjects, an opportunity was also created to explicitly talk about the limit concept as an object.

The next subsection provides information on how the data were collected. These include the size of the sample and the dates on which the data were collected.

Data collection

Questionnaire 1, the limit of a function, was administered on 24th March 2004 to a group of 251 out of 270 first year students. The interviews were held within a period of two weeks after the questionnaire was administered. This follow up

was done within a short space of time so that subjects could still be in a position to remember why they responded to the questions the way they did. For interviews, fifteen subjects were chosen on the basis of their questionnaire responses. In order to be chosen for the interview, one had to have committed errors in responding to a substantial number of questions. The way of responding to the questions among the subjects should also have had some variation. Five subjects were interviewed individually while ten were interviewed in two groups of three and one group of four. All the eight interviews were audio-taped and transcribed by the researcher.

In the second year of study the questionnaire on the limit of a sequence was administered on 23rd October 2004 to 56 subjects out of a class of 70, of whom 14 were repeaters. For interviews, 18 subjects, non-repeaters, were interviewed within a period of two weeks after the questionnaire data were collected. Nine subjects were interviewed individually and the other nine were interviewed in three groups of two and one group of three. Individual interviews took 1 hour each and the group interviews on average took about 1½ hours. Figure 4.1 shows the times on which the data were collected at both the pilot

stage and in the actual study. The data collection took place from the second semester of the 2002/2003 academic year to the first semester of 2004/2005 academic year. As indicated in the introductory chapter, the academic year starts in August and ends in July of the following calendar year. Only the first letters of the months are used in the figure, e.g., the letter A stands for August, S stands for September, etc. The words 'questionnaire' and 'interview' are abbreviated as 'Qst' and 'Int' respectively.

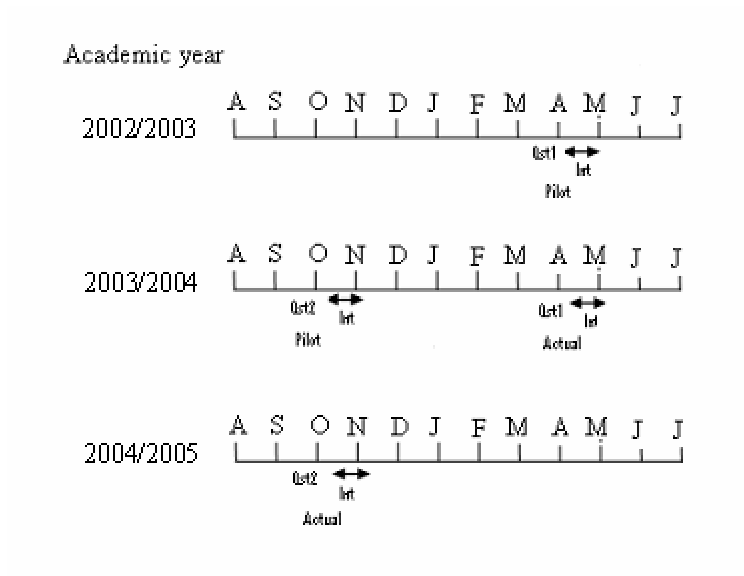


Figure 4.1 Time frame for data collection

To improve the credibility of the results of the study reliability and validity issues were considered.

Reliability and validity

Credibility of the research studies rests on the reliability of their data, methods of data collection, and also on the validity

of their findings (Lecompte & Preissle, 1993; Seale, 1999; Cohen *et al*., 2000; Silverman, 2001).

Reliability

External reliability concerns the replicability of the whole study (Lecompte & Preissle, 1993). The assumption made is that a researcher using the same research methods as the former can obtain the same results as the former (Lecompte & Preissle, 1993; Seale 1999; Silverman, 2001). Internal reliability concerns the degree to which researchers applying similar constructs would match these to data the same way as the original researchers (Lecompte & Preissle, 1993; Seale 1999; Silverman, 2001). A problem with such expectation is the extent to which sets of meanings held by multiple observers are congruent enough to describe and arrive at inferences about phenomena in the same way (Lecompte & Preissle, 1993).

To improve the reliability of research results low inference descriptors (Seale 1999) were used. This involves "recording observations in terms that are as concrete as possible, including verbatim accounts of what people say, for example, rather than researchers' reconstructions of the general sense of what a person said, which might allow researchers' personal perspectives to influence the reporting." (Seale, 1999, p. 148).

In this study the condition of low inference descriptors was satisfied by tape recording all face to face interviews, carefully transcribing the tapes, and presenting extracts of episodes in reporting the results. The purpose served by these extracts is to give the reader or the person who wishes to duplicate the research a better picture of the situation studied in order to make an informed decision in judging the suitability of the analysis.

Other ways of reducing threats to external reliability influenced by the work of Lecompte and Preissle (1993) include provision of information on: researcher status position, informant choices, social situations and conditions, analytic constructs and premises and methods of data collection and analysis.

In this study the relationship the researcher had with the subjects is that the researcher had taught the subjects mathematics at PESP and also was helping with the running of the first year mathematics tutorial sessions. So they are a group that the researcher had known for a reasonable amount of time.

Every subject acted as an informant, that is, through responding to questions on the questionnaires or in interviews. When individual and group interviews were conducted the researcher felt that the atmosphere was very conducive. The researcher started by telling every subject that the focus of the study is not on whether subjects give right or wrong answers but in finding where the conceptions that they possessed about the limit concept could have originated. Having said this, however, does not erase the fact that people are individuals. What may seem to be conducive to the researcher may not necessarily have been so to some subjects, as in most cases there is always some professional distance between students and lecturers. Some subjects may also have felt uncomfortable to give responses they were not fully competent about in the presence of their peers.

Constructs used in the study were those developed throughout chapters one to three, the theoretical framework chapter. Such constructs include: Epistemological obstacles, understanding, language, symbolism, actions, processes, objects, schema and the limit concept. The discussions in the study have shown how the researcher had conceptualised the stated constructs. For those constructs one cannot claim that the interpretations

given are universal. This is because our understanding is influenced by factors such as experiences, beliefs, prior knowledge, etc.

Since both reliability and validity issues had to be considered in improving the credibility of the research results, the next subsection discusses the ways in which validity issues were taken care of in implementing the actual study.

Validity

Internal validity is the extent to which scientific observations and measurements are authentic representations of some reality whereas external validity would mean the degree to which such representations may be compared legitimately across groups (Lecompte & Preissle, 1993). Positivists' view is that social environment has objective reality (Lecompte & Preissle, 1993; Gall M.D, Borg W.R., & Gall, J.P., 1996). This means that social reality is independent of those who observe it, and that observations of this reality if unbiased constitute scientific objective reality (Gall *et al.,* 1996). Constructivists' view is that social environment has constructed reality (Lecompte & Preissle, 1993; Gall *et al.*, 1996). This is because each participant, including the researcher, comes into a study with different set of background experiences, beliefs and values, and each interprets what happens differently. Hence, reality (or validity), can only be achieved in degree.

The validity of the research results was enhanced using the methodological triangulation. This involves studying the nature of the problem from a variety of viewpoints in order to expand the understanding of the phenomenon under study

(Burns & Grove, 1993). If various methods correspond the researcher becomes confident about the findings (Cohen & Manion, 1994). It also helps in the comparison of data. In this study the triangulation was achieved by using questionnaires and interviews. The gaps identified in questionnaires were complemented by data collected using interviews. For example, there are cases in which the subjects' responses did not display obvious errors indicative of epistemological obstacles in their work. But when interviewed, it was revealed that some of the correct answers given were obtained through improper methods of computation for finding limits. The data were analysed soon after collection.

Ethical issues

Some ethical issues taken into consideration when reporting the study include keeping the subjects' names anonymous. The subjects were informed about this. They were informed as members of a class or group verbally before participating in the study. Instead of using the subjects' names, the subjects have been referred to as Subject 1, abbreviated S1, Subject 2, abbreviated S2, etc. Because the questions which were responded to were very many it is likely that even if the subjects themselves read their own work they would not know that they are the ones who gave such answers. This is important in this study because the study was dealing with the type of questions that could be believed to classify the subjects as being good at calculus or not good at calculus. Students' responses are also not marked as right or wrong so when some of them were called for interviews they only knew they were going to be asked questions in relation to the type of responses that they had given and not on how much they had obtained as the responses were not awarded marks.

As the researcher constituted a team of lecturers who conducted the tutorial sessions in calculus during the first year of study for the subjects, one was very careful not to discuss anything with regard to their questionnaire responses as this could have made some subjects feel uncomfortable. The researcher also had a lot of interaction with lecturers in the mathematics department including the lecturer who was teaching calculus, but not even once were students' responses ever discussed. The only information that was ever disclosed to the calculus lecturer was concerning the type of questions that the students were asked in the questionnaire. A sample questionnaire was taken to the lecturer's office a day before the subjects were to respond to the questionnaire. The concerned lecturer was allowed to read and return it to me soon after reading.

Permission to involve the subjects in the study was sought from both the subject lecturer and the subjects themselves. As mentioned earlier the researcher had taught the subjects in the PESP and had very good relations with them. The questionnaire data collected from the subjects is kept in files and the only person who has access to these files is the researcher. The interview audio cassettes are also in a position where they can only be accessed by the researcher. Another ethical issue with regard to professionalism involves writing the limitations of the study and this part appears in section 4.3. The procedures used in data analysis are discussed next.

Data analysis

The data were read over and over again to get an overall picture to the type of responses that the subjects had given. The subjects' errors were first identified from the questionnaire data. This was done by looking for the wrong answers or wrong working from the subjects' scripts. The answers were provided either through working, making choice of answers from the options provided and/or by looking at the

incorrect explanations that did not match their choice of answers. Since these errors constitute epistemological obstacles (Brousseau, 1997), conceptions around which they originate were either identified or inferred.

Bearing in mind the three categories developed from the APOS framework, the data were revisited but this time focusing on one category at a time. The first category was 'epistemological obstacles in interiorising actions into processes'. In identifying data that matched this type of conception, the researcher had to be in a position to differentiate an action from a process, then look for the indicators of the action conception first in the questionnaire responses and later in the interview responses to find out if the interview data supported or contradicted the conception displayed in the questionnaire. Responses of similar nature were grouped under the same category.

For the second category 'epistemological obstacles in constructing coordinated pair of processes', the concentration was on the errors committed in an attempt to coordinate the domain and the range processes via the given function. A sequence is a special type of function. The same process as in the first category was followed. For the third category

‘epistemological obstacles in encapsulating processes into objects’, language markers for the object conception were used as indicators for the object conception of the idea of limit.

The role of language and symbolism was identified by looking at the way the subjects had responded to the questions through their use. For example, it could be by communicating or manipulation of the symbolic surface structures as will be observed in chapter 7.

By looking back at the discussion carried in this chapter, in summary data analysis involved the following:

- Identification of errors in students’ response to the questions;
- Relating the errors to the conception that may have given rise to them;
- Explaining why the identified conception is an epistemological obstacle in the given context;
- Finding the context in which the identified conception is applicable if any; and
- Carrying out the discussions using the theories discussed in the theoretical framework.

As there is no study that can be conducted without any gaps, the next subsection discusses the limitations of the study.

Limitations of the study

There was a high drop out rate for the number of subjects from the first year of study to the second year. This was unavoidable because it is a situation that was determined by the failure rate and also by the subjects' choice of subject of specialisation. Because of this the overall behaviour displayed by subjects in the limit of a function could not be compared to that of the limit of a sequence.

Since the interviews were held within a short space of time after the administering of the questionnaires, subjects chosen for interviews were chosen before a thorough in depth data analysis was done. This situation has resulted in having categories of responses which do not have interviewees in it. It is acknowledged in the discussions on data analysis where such cases are encountered.

Absolute reliability and validity of the study could not be achieved. This is because in this study each participant, including the researcher, came with a different set of background experiences, knowledge, beliefs and values. Hence, the interpretations given have an element of being individualistic in nature.

The results of the study could also not be fairly compared across groups since each group that was studied has its own characteristics different from others where similar studies were conducted. The results of the study can thus not be generalised to the total population of the study except to situations of similar kind.

The main theoretical framework chosen, the APOS theory, concentrated more on the cognitive aspect of behaviour. Hence, in the analysis social aspects of learning did not appear as much as the cognitive. But to obtain some form of balance, the semiotic framework had some social aspects of learning.

As the questionnaires and the interviews were conducted in English, the second language of the subjects, in some cases language issues might have interfered with the subjects' intention to say what they meant or mean what they said. Hence, some interpretation might have been made on unintended meanings.

Since a case study is characterised by the collection of large amounts of data, it has been practically impossible to show all

data to the reader. Thus the results presented may to a certain extent reflect the researcher's choice of data.

This chapter has discussed the different stages in the implementation of the study. The next two chapters, five and six, report on the results of investigating epistemological obstacles within the context of the limit of a function and the limit of a sequence respectively.

Limit of a Function: Results and discussion

As epistemological obstacles in investigating the limit concept were investigated in both the contexts of the concepts of a function and a sequence respectively, this chapter reports on the results of investigating these obstacles within the context of a function. Though a sequence is a function, it is however a special type of function. Hence, there was a need to investigate the idea of limit in both contexts. While a function may be defined over an interval, and therefore its graph be drawn by a curve, a sequence is made up of discrete points which cannot be joined by a line. This is because the domain of a function could assume any real number whereas the domain of a sequence consists of counting numbers only. The members of the domain of a sequence represent the position of the terms of the sequence and this is the reason why they can only be counting numbers.

The background knowledge of this group of students, in finding the limits of functions, includes finding the limits of ordinary functions, piece wise functions and composite functions. The most popular representations encountered being the numerical and the algebraic. However, the recommended calculus text book by Finney and Thomas has the geometrical modes of representation. The subjects will have used the tabular method, the substitution method, and the algebraic methods such as rationalising before the limit value can be obtained through the limiting process of 'tending to'. Application of the chain rule and L'Hospital's rule will also have been encountered together with the indeterminate forms of limits. In this study the questions set involved the four modes of representation, the numerical, the algebraic, the geometrical and the descriptive. As already mentioned in chapter 1, this group of students did not cover the formal definition of limit. Hence, the questions were responded to with the knowledge of the informal definition of limit only.

Each of the 251 subjects who constituted the sample of the study responded to the questionnaire. Fifteen of these subjects were interviewees. The presentation that follows relates to the categories from the APOS framework. These are:

- Epistemological obstacles in interiorising actions into processes;
- Epistemological obstacles in constructing the coordinated processes; and
- Epistemological obstacles in encapsulating processes into objects.

As mentioned in the methodology chapter, one of the limitations of the study is that one had to deal with a huge amount of data. Hence, it has not been possible to show all these data in discussing the results. The discussions in chapter 5, 6, and 7 will therefore present extracts of subjects who are a representation of the other subjects in the same category.

Epistemological obstacles in interiorising actions into processes

The actions considered in this section will be applied on data from questionnaires 1 and 2. They are described in research by the first two steps by Cottrill *et al.* (1996):

1. The action of evaluating f at a single point x close to or equal to a.
2. The action of evaluating the function f at a few points (or any finite number), each successive point closer to a, than was the previous point.

A subject who displayed these steps in their work had the action conception of the limit concept. A process schema in this case is constructed if there is a movement from the mental construction in step 2 to mental construction in steps 3(a) and 3(b):

3(a) Interiorisation of the action of step 2 to construct a domain process in which the independent variable x approaches a; and

3(b) The construction of the range process in which y the dependent variable approaches L.

Questions whose results will be used to show the stated mental constructions are Questions 4 and 6 from questionnaire 1. This is because in answering these questions the relevant mental constructions stated by Cottrill *et al.* were visible in students' work. The presentation will focus on one question at a time. In each case the question is presented. Part of the schema that was to be applied in responding to specific parts of the questions is also stated. Table 5.1 gives a summary of results.

Question 4: How can we see if a function $y = f(x)$ has the limit L as x is approaching 0? It is by:

1. Calculating y for $x = 0$, i.e., calculate $f(0)$;
2. Calculating $f(1)$, $f(2)$, $f(3)$ and so on and observe the results;
3. Calculating $f(x)$ for $x = 1/2$, $1/4$, $1/8$ and so on;
4. Substituting x by 0 in the function formula, and calculate the value;
5. Substituting numbers that are very close to 0 for x in the formula and look for the value of y.
6. Substituting numbers that are very close to 0 for x in the formula and look for the value of y that is being approached as x values approach 0.

Choose the option(s) that best describes your answer.

Why will you do so?

Part of the schema that had to be applied in responding to this question was:

- The construction of the domain process $x \to 0$;
- The construction of the range process $f(x) \to L$; and
- Coordination of the domain process and the range process to obtain the limit value.

Subjects who have been identified as having action conception are those who chose options 1, 4, or 5. This is because a choice of these options considers either one computation (options 1 and 4) or a finite number of computations (option 5). Option 2 reflects the domain process $x \to \infty$ rather than $x \to 0$. Option 3 also reflects the process conception, indicative in the words 'and so on' which considers an infinite number of computations. In both options 2 and 3 there is no coordination between the domain process and the range process to yield the limit value. Hence, they cannot be the correct options. Option 6 is the correct option as it reflects the construction of a coordinated process schema.

Table 5.1 Results of responding to Question 4

Option chosen The limit value is calculated by:	Number of subjects	Number of subjects in %
1. Calculating y for $x = 0$.	19	8
2. Calculating $f(1)$, $f(2)$, f(3) and so on and observe the result	2	1
3. Calculating $f(x)$ for x = ½, ¼,1/8, and so on	2	1
4. Substituting x by 0 in the function, and calculate the value	29	12
5. Substituting the numbers that are very close to 0 for x in the formula and look for the value of y.	20	8
6. Substituting the numbers that are very close to 0 for x in the formula and look for the value of y that is being approached as x values approach 0.	120	48
More than one option	43	17
No response	17	7

The next subsections discuss the epistemological obstacles that were encountered by students for making the specified choice of options.

1.1.1 Option 1: Calculating y for x = a.

For the subjects who chose this option, four of them did not give reasons for their choice of answer. Of the remaining fifteen subjects, 11 subjects gave reasons that displayed action conception. These are:

- The limit value is obtained by the substitution method (10 subjects);
- The limit exists if the function is defined (1 subject);

The limit value is obtained by the substitution method

The interview extracts that follow represent the type of reasoning that the 10 subjects in this category gave:

> S16: When x in the $f(x)$ is substituted by zero, y value which corresponds to zero is L $(0,L)$ will be found to be the limit.
>
> S232: The limit of $f(x)$ as x approaches zero when taking $f(x)$ to be $f(0)$ we will get L our limit value.

Substitution method is a method that is found in most calculus texts and it is used in the computation of the limit values by considering an x that is equal to a, an action. Though this method produces the correct limit values for continuous functions, it obscures the students' chances of understanding the limit concept. This is so because the symbolism used, $x \to a$, does not in any way suggest that at some point x is equal to a will ever form part of the computation. Tall (1996, p.305) also says: "… in terms of considering the expression $\lim_{h\to 0}\frac{(x+h)^2 - x^2}{h}$,…. The fact that the simplification can be done for $h \neq 0$, yet to obtain the limit one puts $h = 0$, in the expression $\lim_{h\to 0}(2x+h)$, also contributes to the conflict." Since students were concentrating on only one evaluation at $x = a$, clearly there was no evidence for the interiorisation of an action into a process. The stem of the question in this case explicitly referred to the use of the informal definition of the limit. But having used the substitution method, students think that the method is applicable to every function. Thus over-generalising is an epistemological obstacle.

The limit exists if the function is defined

The response of S162 was as follows:

S162: So as to see if the $f(x)$ is defined at $x = 0$.

S162 associates the existence of the limit with whether or not the function is defined at a point. Thus the subject confuses the function value with the limit value. The function value does not exist where the function is not defined. But the limit value may exist even where the function is not defined. Thus generalising this conception to the existence of the limit value is inappropriate. This generalisation may result from the type of functions which the students may have encountered in class. For example, the function, $f(x) = \frac{1}{x}$, is not defined for x=0 and the limit for the function does not exist not because it is not defined for x=0, but because its function values tend to ∞ as x values tend to 0. This response reflects an action conception because a single evaluation of f at $x=a$, is considered as was the case in using the substitution method.

Option 4: Substituting x=a in the formula

This option is the same as option 1. They only differ in the way they have been presented. Out of 29 subjects who chose this option, ten of them did not give reasons for their choice of answer. The other subjects gave reasons which originated from the following conceptions:

- The limit value is obtained by substitution method (18 subjects); and
- The limit value is obtained by the points in the neighbourhood (1 subject).

The limit value is obtained by substitution

Some of the responses given by the subjects are:

S102: It is because for any given function, a number given for x is substituted in the formula to get the value of the limit.

S133: Because that is the only way to find the limit of a function.

> S97: Because we have a simple substitution method to find whether the limit exits or not.

As suggested earlier, substitution method displays an action conception stated as the first step by Cottrill *et al*. This method has some contradictions with the dynamic definition of the limit concept, $f(x) \to L$ as $x \to a$. Its use is encouraged by the way continuous functions are defined at some number a, $\lim_{x \to a} f(x) = f(a)$. Students use this method to obtain correct answers. To say that the function value is equal to the limit value does not mean that the function value is the limit value. The function $\frac{\sin x}{x}$, for example, has the limit value 1 as x tends to 0. But the function value does not exist at $x = 0$. Probably students make generalisations about the applicability of the substitution method because they probably cannot think of functions in which substitution method cannot work. Over-generalising therefore acts as an epistemological obstacle when used in inappropriate contexts.

The limit is determined by the points in the neighbourhood

The extract of S29 is now presented:

> S29: Because we are interested to find what happens to the functional values in the neighbourhood of the value of *x* of the function.

Though the interiorisation of the actions into processes had started here, by considering a series of values, interiorisation of actions into processes is not fully developed either into the domain process or the range process. This behaviour still has a static feel. There is no consideration of any values tending to another value, the limit value.

Option 5: Substituting the numbers that are very close to a

Five subjects out of 20 subjects who chose this option did not give reasons for their choice of answer. Some justifications for choosing option five by the subjects were:

- When finding limits we are interested in the neighbourhood of a point (14 subjects)
- In finding limits we have to find its indeterminate form (1 subject).

When finding limits we are interested in the neighbourhood of a point
Some of the responses within this category were:

> S173: Because I do not want the limit when x is zero but I want the limit when x is any number very close to zero, either from left or from the right.
>
> S196: As to see what happens to the value of y when x is approaching 0 from the negative and x is approaching 0 from positive.
>
> S203: Since x approaches 0 but it is not exactly 0, we substitute it with numbers very close to 0 and look for the value of y.

Here the subjects take the limit value to be the function value. Such function values are considered to be a result of points in the neighbourhood of zero. This is done by a consideration of a finite number of computations. The subjects seemed to have an incomplete conception of the domain process and the corresponding range process. They were aware that a series of numbers had to be substituted, a beginning of the process conception. But such numbers were not considered up to the level of $f(x)$ tending to L as x tends to a.

In finding limits we have to find its indeterminate form
An extract showing the reasoning given by S66 now follows:

> S66: To look whether $\frac{\infty}{\infty}$ or $\frac{0}{0}$.

$\frac{\infty}{\infty}$ and $\frac{0}{0}$ are indeterminate forms of limit obtained by finding the limit values of both the numerator and the denominator of the quotient function. S66 suggests that one cannot find the limit value unless these states are established. It is not necessarily the case that every function will go through the indeterminate state before its limit value is computed. This misconception acts as an obstacle to knowledge acquisition as subjects seem to be ruled by calculations or procedures instead of the meaning inherent in the calculations.
The next part discusses the results of responding to Question 6. In responding to Question 6, the action conception was made visible in students' work in responding to Question

6(iii). The discussion in this subsection will therefore be based on the results of this question only. The results for Questions 6(i) and 6(ii) are presented and discussed in chapter 7 under the role of language and symbolism as they are more related to issues of language and symbolism.

Question 6 (iii): Study the given expression and calculate its limit:

$$\lim_{x \to \infty} \frac{\sqrt{x^2+9}-3}{x}$$

Part of the schema that could be applied in responding to question 6 (iii) was:

- Awareness that the result of substituting gives $\frac{\infty}{\infty}$, an indeterminate form of limit. Hence alternative methods could be used;
- Awareness that if the function is treated as a product $\lim_{x\to\infty}\frac{1}{x}\cdot\lim_{x\to\infty}(\sqrt{x^2+9}-3)$, will give 0.∞, an indeterminate form of limit. Hence, an alternative method could be used;
- Realisation that part of the function to be differentiated $\sqrt{x^2+9}$, is composite. Hence, the chain rule has to be used;
- As part of the chain rule schema, one should have both the process and the object conception of a function. They should also see the chain rule as an extension of the power in that both the inner and the outer brackets in $(x^2+9)^{\frac{1}{2}}$, an equivalence of $\sqrt{x^2+9}$, have to be differentiated and the results multiplied by each other; and
- Coordination of the domain and the range processes to obtain the limit value.

Table 5.2 Results of responding to Question 6 (iii)

Question	Response	Number of subjects	Number of subjects in %
Calculate $\lim_{x\to\infty}\frac{\sqrt{x^2+9}-3}{x}$	1	89	35
	∞	43	17
	0	33	13
	$\frac{\infty}{\infty}$	10	4

This question required the subjects to compute the limit value. The next subsection discusses epistemological obstacles that were encountered in responding to this question.

Epistemological obstacles in computing the limit value

In responding to this question, 47 subjects displayed the action conception. Five of these subjects were among the subjects who obtained 0 as the answer, 8 were among those who obtained ∞ as the answer and 34 were among the ones who obtained one (1) as the answer. Their responses reflected the existence of the mental constructions stated by the first two steps of Cottrill *et al*. Some subjects used the substitution method and some used 2 or 3 values in the neighbourhood of $x = a$ in the computations before reaching a conclusion about the limit value.

Using one value at *x* = a in computing the limit

This is step one of the action conceptions by Cottrill *et al* (1996). Extracts showing how the different answers were obtained are presented:

S67: $$\lim_{x\to\infty}\frac{\sqrt{(1x10^6)^2+9}-3}{1x10^6}=\frac{1x10^6+3-3}{1x10^6}=1$$

S136: $$\lim_{x\to\infty}\frac{\sqrt{x^2+9}-3}{x}=\lim_{x\to\infty}\frac{(\sqrt{(x)^2+9}-3)(\sqrt{x^2+9}+3)}{x\sqrt{x^2+9}+3}=\lim_{x\to\infty}\frac{(x^2+9)-9}{x\sqrt{x^2+9}+3}$$

$$=\lim_{x\to\infty}\frac{x^2}{x\sqrt{x^2+9}+3}=\lim_{x\to\infty}\frac{x}{\sqrt{x^2+9}+3}=\frac{\infty}{\sqrt{\infty^2+9}+3}=\frac{\infty}{\infty}=1$$

S51: $$\lim_{x\to\infty}\frac{\sqrt{x^2+9}-3}{x}=\frac{\sqrt{(1\infty)^2+9}-3}{\infty}=\frac{\infty}{\infty}=\infty$$

S227:

$$\lim_{x\to\infty}\frac{\sqrt{x^2+9}-3}{x}$$
$$=\lim_{x\to\infty}\frac{1}{2}x2x(x^2+9)^{-\frac{1}{2}}$$
$$=\lim_{x\to\infty}x(x^2+9)^{-\frac{1}{2}}$$
$$=\frac{1}{\sqrt{x^2+9}}=\frac{1}{\infty}=0$$

In simplifying the radical part S67 writes: $\sqrt{(1x10^6)^2+9}=1x10^6+3$. This is a wrong simplification of the radical. It is equivalent to saying that $\sqrt{a^2+b^2}=a+b.$ This simplification was performed by 32 students in this class. It is a point which will be discussed under the role of language and symbolism in chapter 7. Having met simplifications such as $\sqrt{a^2b^2}=ab,$ it is now generalised to an inapplicable context. S136 performed the computation $\frac{\infty}{\infty}=1.$ The computation is similar to that performed by one of the subjects in Sierpinska's (1987) class. The subject explained this as an interpretation that the number divided by itself is one. S51 was among the eight subjects who performed the computation $\frac{\infty}{\infty}=\infty,$ which could be interpreted as: a bigger number divided by a bigger number produces a bigger number. S227 carried out the computation $\frac{1}{\infty}=0.$ This conception is similar to the one which was possessed by Wallis and his contemporaries in the past. It now appears in the educational practice today.

Computation using 2 or 3 values in the neighbourhood of *x* = *a*

This is step 2 of the action conceptions by Cotrill *et al.* Two extracts showing how this method was used now follow:

S27:

x	(- 1x10^{35})	(-1 x 10^{15})	(1x10^{15})	(1x10^{35})

lim	1	1	1	1

$$\therefore \lim_{x\to\infty^-} \frac{\sqrt{x^2+9}-3}{x} = 1$$

$$\therefore \lim_{x\to\infty^+} \frac{\sqrt{x^2+9}-3}{x} = 1$$

S49: $\lim_{x\to\infty} \frac{\sqrt{x^2+9}-3}{x}$

$$= \lim_{x\to\infty^-} \frac{\sqrt{x^2+9}-3}{x} = 1$$

The tabular representation that follows constituted part of the work by S49:

10	100	1000	10000
0.7	0.97	0.997	0.9997

S27 perceives ∞ as something that can be approached from both the left and the right. Here the subject treats negative infinity ($-\infty$) the same as approaching infinity from the left (∞^-). S27 infers the characteristic properties of numbers to the concept ∞, by generalising. In every computation S27 has rounded the answer to the nearest whole, which is one in this case. This value is then taken as the limit value. S49 is aware that there is only one way of tending to ∞. That is, by a consideration of large positive numbers. It is however not clear from the work shown whether the answer S49 obtained was through the limiting process of tending to or by rounding off.

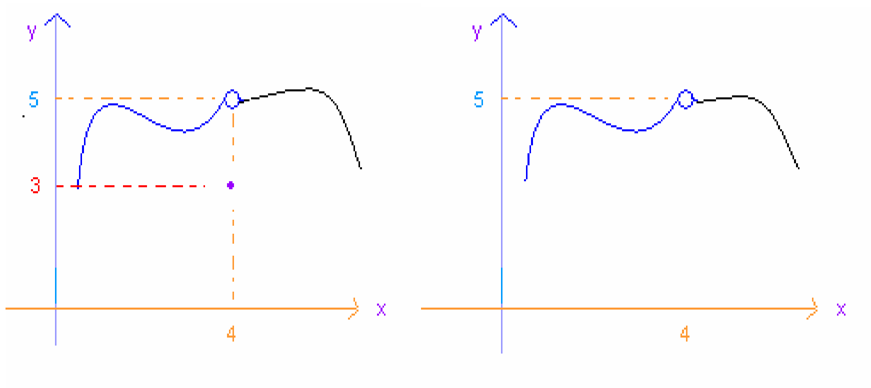
Epistemological obstacles in constructing a coordinated pair of processes

The coordinated pair of processes schema considered here is $f(x) \to L$ as $x \to a$; classified as step 3 (c) by Cottrill *et al.* (1996):

3(c) Coordination of mental constructions in steps 3(a) and 3(b) via *f*. That is, the function *f* is applied to the process of approaching *a,* to obtain the process of *f*(*x*) approaching *L*.

Questions whose results will be considered in this section are Questions 3 and 5. Question 3 deals with the limit concept in the geometrical or graphical mode while Question 5 has the numerical, the algebraic and the descriptive modes of representation. Question 3 is first presented with its results and discussion. These will be followed by Question 5.

Question 3: You have been asked to find the limits of the functions below as x tends to 4 (if any). Which of the statements below would you agree with about the two functions?



(i) **(ii)**

(a) **In diagram (ii) the limit does not exist since the function is not defined at $x = 4$.**

(b) **In diagram (i) the limit is 3 since it is the function value at $x = 4$.**

(c) **The two functions have the same limit since we are not concerned as to what happens at $x = 4$ but to function values in its neighbourhood.**

(d) **The limits for the two functions cannot be obtained since the two functions are not defined at $x = 4$.**

Why do you agree with the statement(s) you have chosen?

This question relates the students' interpretation of a function with the conventional symbolic concepts imbedded. Part of the schema that was to be applied includes:

- Awareness that the first function is defined at $x = 4$ and the second is not.
- Awareness that the function value of the first graph at $x = 4$ is 3.
- Awareness of numbers in the neighbourhood of $x = 4$;
- Construction of the domain process in which x approaches 4;
- Construction of the range process in which $f(x)$ approaches 5; and
- Coordination of the domain process and the range process via f in geometric mode of representation to yield $f(x)$ approaches 5 as x approaches 4.

The results of responding to Question 3 are presented in Table 5.3.

Table 5.3 Results of responding to Question 3

The option(s) chosen	Number of subjects	Number of subjects in %
(a) The limit does not exist where the function is not defined	10	4
(b) The function value is the limit value	14	5.6
(C) Neighbourhood of $x = a$, determines the limit value (the correct option)	142	57
(d) The limit does not exist where the functions are not defined	25	10
More than one option	29	12
No response	31	12

The discussions that follow show the type of epistemological obstacles that were encountered in choosing the stated options.

1.1.1 Option (*a*): The limit does not exist where the function is not defined

For this option, 3 subjects out of 10 subjects did not give reasons for their choice of answers. Those who did seemed to commit errors related to the following conceptions:

- The limit exists where the function is defined (4 subjects);
- If $f(x)$ is not defined at a point the function values go to infinity (1 subject);
- A piecewise function has two limits (1 subject); and
- Limit values can be obtained only if different functions are used (1 subject).

The limit exists where the function is defined

The excerpts that follow show the reasons that the subjects gave for choosing option (a) as the answer.

S4: Because if y as 5 is substituted in the function there is not value of x. Therefore the limit of the function does not exist at $x = 4$.

S68: We are concerned as to what happens at $x = 4$ but in diagram (ii) that concern is not indicated.

S71: Because at x=4 we can be able to calculate the change in x, hence be able to use the expression: $\lim_{x \to 4} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$.

S99: At x=4 the presentation (–o—) means x=4 is not contained in the function.

S4 confuses the domain with the range. Instead of saying that the x value is substituted to get the y value the subject refers to the y value as the one that is being substituted in order to get the x value. As pointed earlier, while the existence of the function value depends on whether the function is defined at a particular point or not, this is not necessarily the case with the limit value. The limit may exist even where the function is not defined. For instance, in diagram (ii) the function is not defined at $x = 4$, but the limit exists because $f(x)$ tends to 5 as x tends to 4. Thus subjects here seem to consider what happens at $x = 4$ and not what happens as x values tend to 4. S68 explicitly says that we are concerned to what happens at $x = 4$. S71 bases his computation of the limit value on using the formula. The symbolism on the right hand side would be used in the computation of the gradient. The symbolism on left hand side does not make any sense at all. What the subject wanted to write was perhaps the symbolism used in the computation of the derivative as $\lim_{\Delta x \to 4} \frac{f(x+\Delta x) - f(x)}{\Delta x}$. The subject does not seem to know when this formula is used.

If $f(x)$ is not defined at a point the function values go to infinity

The extract showing this conception is presented:

S223: At $x = 4$ the two functions are undefined. They go to infinity.

If we had the function, $f(x)=\frac{1}{x}$, for example, and were to find its limit as x tends to 0, we would show that the limit does not exist by writing $f(x)\to\infty$ as $x\to 0$ or $\lim_{x\to 0}\frac{1}{x}=\infty$. This is so not because the function is not defined at x = 0, but because as x tends to 0, $f(x)$ values tend to infinity. So having met an example of this kind, a subject makes a generalisation that if a function is not defined at a particular point it means that its $f(x)$ values tend to infinity. Thus an irrelevant property is generalised.

A piece-wise defined function has two limits

S85 wrote:

> S85: The limit does not exist because there is more than one value of limits of this function, 3 and 5.

In the first graph 3 is the function value at x = 4 and 5 is not. However, as x tends to 4, the function values tend to 5 and not 3. So, one cannot refer to the function value 3 as the limit value. A conclusion about the limit value in this case is based on inappropriate information. The two functions have the same limit since they both tend to 5 as the x values tend to 4. Representation here seems to be an epistemological obstacle. A graph is said to represent a function only when it is shown by a continuous line. A point that stands on its own is not taken to be part of the graph. One would expect that subjects will take the graph of a sequence as incomplete since it is made up of discrete points which should not be joined. This also relates to how the subject interprets the graph. The hole or ring symbol on the graph carries a special meaning which is not fully understood by the subject.

In finding limit different functions have to be used

S8 gives the following response:

> S8: Because the two functions are the same hence the limit of each cannot be obtained.

This seems to be a familiarity problem. The subject takes the two functions to be the same while they are not, both on the surface and at a deeper level. The first function is defined at x=4 while the second is not. The subject finds it rather an unusual situation to be asked to find the limits of the same functions. Hence, he draws a conclusion that the limit of each function cannot be obtained. Perhaps what the subject means here is that the two functions should have the same limit since the graphs representing them are the same. So, finding the limit of each as though they are different is not possible.

Option (b): function value is the limit value

For option (b), six subjects out of 14 subjects did not give reasons for the choice they made. Three conceptions stated below were inferred from the reasons that led the students to choosing option (b).

- The limit value is the function value (5 subjects)
- The limit is the boundary (1 subject);
- The formula determines the existence of a limit (1 subject); and
- The asymptote determines the existence of the limit of the function (1 subject).

The limit value is the function value

The extracts now follow:

S44: Because the limit at $x = 4$ can be defined at $y = 3$.

S114: Because at $x = 4$ the limit is not defined as the graph has indicated by 'o' not shaded.

S156: Because the (i) graph shows that the limit exists and it tends to x is shown by the graph which give the value 3.

S180: According to the graph it shows that 3 is the function value at x =4.

S129: A function has a domain and a range, $4 \rightarrow 3$.

As already stated, for a continuous function, the function value and the limit value assume equal values. But conceptually the two are different. We can only say that the function value and the limit value are equal and this does not mean that we can confuse the function value for the limit value. In diagram (i) 3 is the function value for $x = 4$ and 5 is not. When approached from both sides, the function values approach 5 as the limit value. In a study by Cottrill *et al.* (1996), subjects also interpreted the limit of a function at a point as the function value at that point. One consequence of this conception is that the existence of the limit where the function is not defined will be denied.

The limit is the boundary

The extract of S97 now follows:

> S97: Because we have initial and final values of the graph along the y-axis and we have initial and final values along the x-axis which make it possible for one to find the limit of the graph.

The everyday meaning of the word ‘limit’ influences the subject’s response. The subject says that the limit of the graph can be found because the initial and the final values which set the boundary for the function are given. And in this case these initial and the final values mark the limit. That is, the graph does not go beyond these points. In a study by Davis and Vinner (1986) the subjects confused the limit with a bound. They said that the limit has to be an upper or a lower bound for all a_n in the sequence. In Frid’s (2004) study the subjects also referred to the limit as a border than cannot be crossed.

The formula determines the existence of a limit

The answer that S126 gave is presented:

> S126: Because the function that will clarify is not given.

The limit value for a function in any mode of representation can be obtained. Here the subject thinks that this can only be possible if the graph of the function is accompanied by its algebraic representation. It is not possible to translate every graphical representation of a function to its algebraic mode, which means that in such cases a subject may also think that it is not possible to find the limit value. S126 was an interviewee. His reason did not match his choice of answer. But based on the option chosen this is what he had to say in an interview:

R: Is 3 the number that $f(x)$ is approaching as x tends to 4? Here when we are finding the limit as x tends to 4, are we interested in what happens as $x = 4$ or are we interested in what happens in the neighbourhood of 4?

S126: We are interested with what happens in the neighbourhood of 4.

R: So does it matter whether the function is defined at 4 or not?

S126: No, it doesn’t matter.

R: So, with your new understanding, can you tell me as I come closer and closer to 4, what value is this function approaching?

S126: It is approaching 5.

R: So is 5 the limit or not the limit of this function as x tends to 4?

S126: It is the limit of this function.

His option reflects the conception that the limit exists where the function is defined. He overcomes this obstacle by realising that the existence of the limit value for this point does not depend on whether or not a function is defined at that point. The subject's change of mind might have been influenced by the question "Does it matter whether the function is defined at 4 or not?" The question seemed to be leading the subject in some way. Thus the subject seems to improve on his conception of the limit when asked the stated questions.

The asymptote determines the existence of the limit of a function
The extract that follows reflects this conception:

S216: The limit is the point where the curve will definitely have to stop, limit of a curve is simply an asymptote, a curve cannot cross the asymptote unless $r(x) = 0$.

Everyday meaning of the word limit as a point beyond which one cannot go, acts as an obstacle here. The subject refers to the limit as a point where the curve will have to stop. In dealing with functions such as $f(x) = \frac{1}{x}$ as x tends to either 0 or ∞, asymptotes can be used to determine the limit values. But in this question the idea of asymptote is irrelevant as there are none.

Option (c): Points in the neighbourhood of x = a determine the limit value

Though option (c) was the correct option for the question asked, there are still some subjects who chose this option for inappropriate reasons. That is, reasons originating from conceptions which acted as epistemological obstacles. Of the 142 subjects who chose this option, only 77 of them gave reasons for their choice of answer. The reasons were related to the following conceptions:

- Same functions have the same limits (20 subjects);
- Points in its neighbourhood determine the limit point (55 subjects);
- The limit values are critical points (2 subjects)

Same functions have the same limits

Twenty subjects said that the two functions have the same limits on the basis that they are the same. Four subjects (S22, S112, S194, and S244) said that the graphs of the functions were the same without specifying how they were the same. Six subjects (S42, S64, S88, S96, S106, and S232) said that the limits of the two functions were the same based on shape of their graphs. Four subjects (S41, S55, S75, and S248) said that the functions were both not defined at $x = 4$. The last group of 4 subjects (S53, S122, S181, and S187) said that the limits were the same because the functions have the same point (4, 5) and the last group of two subjects (S34, S199) chose the option based on the assumption that the two functions have the same equation. The excerpts that follow show the answers given by one member of each group:

> S22: Because the graphs are the same.
>
> S88: It is because the graphs have the same shape to prove that their limits are the same.
>
> S248: Because the graphs are not defined at $x = 4$.
>
> S181: This is because when $x = 4$ the function shows that $y = 5$.
>
> S34: Because the graphs have the same function that describes them, hence they should have the same limit.

Subjects who considered the two functions the same based on shape of their graphs ignored the fact that the first function is defined at $x = 4$ while the second is not. This point is shown by a dot which is visible. S181 was among the subjects who considered the two functions to be defined at $x = 4$ and having the y value of 5 in each case. Five is not a y value for any of the two functions. All the subjects in this group seem to have problems with interpretation of graphs of functions. In particular, the graphs with a hole or a circle symbol.

Points in the neighbourhood determine the limit value

Of the 55 subjects who based their answers on the neighbourhood of a point, some made explicit statements and others made implicit statements. For implicit statements the key words that were used are 'very close', 'negative and positive sides', 'less than or greater than' and 'left or right'. Extracts that follow show answers falling within the stated categories:

S252: Because talking of a limit we mean the number very close to the targeted one.

S89: Because from side view it is obvious that approaching $x = 4$ from negative or positive side the limit is the same, it is 5.

S30: It is because we are talking about #'s that are less than 4 but very close to it and greater than 4 and get very close to it.

S18: It is because when x approaches 4, one can take values very close to 4, either from left of 4 or from right.

Two subjects, S118 and 191, said that their choice was based on the way the limit is defined, an answer which does not display the actual conception they had. They were however classified with some benefit of doubt under neighbourhood category. The majority of the subjects still do not take the limit as a coordinated process. Their explanations are given in terms of the domain process only. Though the neighbourhood category was the most appropriate choice, the subjects concentrated only on the points in the neighbourhood of 4. They did not mention that the behaviour of corresponding y values also had to be considered. Finding the limit value was not seen as a coordination of two processes, the domain and the range processes.

The limit values are critical points

Two subjects gave the following responses:

S63: The curve is at its maximum at $x = 4$.

S120: Because the critical point occurs whenever the graph changes its concavity increasing or decreasing at some points.

Everyday meaning of the word limit gets into the way of S63. The subject uses the word maximum to refer to the highest point beyond which there is no other point. The subjects are however not aware that the corresponding y value to $x = 4$ is 3. Hence at $x = 4$ we have the coordinates as (4, 3). This point is not the maximum point. There is no corresponding value to $x = 4$ on the second function. Hence, there is no point where the subject refers to as the critical point. Cornu (1991) provides the word 'maximum' as one of the spontaneous models of 'limit'. In this case this conception is revealed by S63.

Option (d): The limit does not exist where functions are not defined

Seven subjects out of 25 subjects did not give reasons for their choice of answers. Errors committed by eighteen subjects who gave reasons seemed to originate from the following epistemological obstacles:

- The limit exists where the function is defined (15 subjects); and
- The formula determines the existence of the limit (3 subjects).

The limit exists where the function is defined

Some of the conceptions from which the stated conception was inferred now follow:

S52: It is because at $x = 4$ the limit of the two graphs is undefined as it has been symbolised by the open symbol i.e. —o— and not —•—.

S117: Because the graphs are not defined at $x = 4$.

S144: Because at $x = 4$ the function is undefined therefore it is impossible to find the limit when $f(x)$ is undefined.

S209: This is because at a vertical asymptote the function is not defined and $x = 4$ is a vertical asymptote.

S236: Because the two graphs are discontinuous at $x = 4$.

S131: This is because both lines where $x = 4$ they are circular meaning they are both undefined.

As mentioned earlier, existence of a limit at a point does not depend on whether or not the function is defined at that point. S52 refers to the two functions as undefined at $x = 4$. But the first function is defined at $x = 4$. S209 makes a wrong generalisation that wherever a function is not defined, a vertical asymptote for its graph exists. The graph of the function $y = \frac{1}{x}$ has a vertical asymptote $x = 0$. This function is not defined at $x = 0$. The graph in this question has no vertical asymptote. S236 on the other hand seems to refer to the everyday meaning of continuity. That is, the graphs of the two functions are discontinuous because they both have breaks at $x = 4$. As already discussed in the mathematical context continuity at a point *a,* is determined by the equality of the function value and the limit value at that particular point and not a break.

The formula determines the existence of the limit
The extracts that follow reflect the stated conception:

S152: Because the graphs have been given and the functions are not stated.

S155: The function is not given.

S237: The functions of the graphs are not given.

This category has also appeared in section 5.2.2. It is a conception that seems to emerge from a classroom practice. The mathematical tasks that subjects interact with are likely to make them believe that the algebraic representation is the only situation in which limits could be obtained. The next subsection presents the results of responding to Question 5 which required translation between modes of representation.

Translating between different modes of representation of the same function

The second set of results is concerned with subjects' responses to Question 5. The question deals with translating between different modes of representations of the same function. Different schemas had to be applied in responding to different questions. Part of the schema that was to be applied in responding to Questions 5(a)(i) and 5(a)(ii) was:

- Recognition that a limiting process of 'tending to' produces a recurring decimal;
- Identification of the rational equivalence of a recurring decimal;
- The construction of the domain process in which x approaches 0.75; and
- The construction of the range process in which $f(x)$ approaches 1.9.

Part of the schema to be applied in responding to Question 5(b)(i) was:

- Translation from the numerical representation of the domain process to algebraic mode;
- Translation from the numerical representation of the range process to algebraic mode;
- Coordination of the range process, ($f(x) \to L$), and the domain process, ($x \to a$), to form the coordinated pair of process, $f(x) \to L$ as $x \to a$;
- Recognition that $\lim_{x \to a} f(x) = L$ is an alternative symbolism to $f(x) \to L$ as $x \to a$; and
- Coordination of $f(x) \to L$ as $x \to a$ with $\lim_{x \to \ldots} f(x) = \ldots.$, where L = 1.9 and a = 0.75, that is, having to know where to fit in the values 1.9 and 0.75 in the expression.

Question 5: The function $y = f(x)$ is calculated for values of x, and here are some results

x	$y = f(x)$
0.7	**1**
0.74	**1.8**
0.749	**1.89**
0.7499	**1.899**
0.74999	**1.8999**

(a) If this pattern continues what can you say about:

(i) The number $f(x)$ is approaching?

(ii) The number x is approaching?

(b) Complete the expression below so that it is true about the function represented by the table of values above.

(i) $\lim_{x \to \ldots} f(x) = \ldots.$

(ii) After completing the expression above, write in words the meaning of the expression.

Part of the schema that was to be applied in responding to Question 5(b)(ii) was:

- Recognition that $\lim_{x \to a} f(x) = L$ has a different descriptive representation to its equivalence $f(x) \to L$ as $x \to a$; the first reads: "limit of $f(x)$ as x tends to a, is L" and the latter reads: "$f(x)$ tends to L as x tends to a".

Out of 251 subjects who responded to the questionnaire, only 222 responded to questions 5a(i) and 5a(ii). These questions required the subjects to construct both the range and the domain processes. Table 5.4 gives statistical results of subjects' responses.

Table 5.4 Results of responding to Question 5(a)(i) and 5(a)(ii)

Pairs of values that $f(x)$ and x are approaching respectively.	Number of subjects	Number of subjects in %
2 and 1	88	35
1.9 and 0.75	60	24
2 and 0.75	20	8
2 and 0.7	12	5
2 and 0.8	8	3
∞and ∞	8	3
2 and 0	4	1.6
1 and 0.7	4	1.6
1 and 1	4	1.6
1.9 and 0.8	3	1
1.8 and 0.7	3	1
1.9 and 1	2	1
Individual responses such as: 1.8 and 0.74, ∞ and 0, ∞ and 1, …	6	2.4

Epistemological obstacles in constructing the domain and the range processes

In responding to Question 5a(i) and 5(a)(ii), conceptions that acted as epistemological obstacles are:

- Limit values are whole numbers (88 subjects); and

- Number of decimal places determines the limit value (11 Subjects); and
- Approximating or rounding off are the limiting processes (3 subjects).

Limit values are whole numbers

In responding to Question 5(a)(i) the popular answer was $f(x)$ approaches 2 and x approaches 1. S127, an interviewee, was among the subjects who gave the stated responses. The extract that follows shows how the subject found 2 and 1 as the answers

The discussion with S127:

R: In your symbols you show that as x approaches 1, $f(x)$ approaches 2. Let us look at this table of values can you add five more numbers in both the column of x and that of $f(x)$. Which number is x approaching? Which number is $f(x)$ approaching? In both cases you say the number is constant. Can you explain what you mean by this?

S127: I realised that as the pattern continued this number is very close to 2.

R: So here we have 1.8999 what do you think the next number would be?

S127: It will be 1.89999.

R: When you think of a number are you thinking of whole numbers only or you also think of fractions or decimals?

S127: I think only of whole numbers.

R: But is it only whole numbers which are numbers?

S127: (Silent)

R: Which number are the numbers on the right approaching?

S127: Two.

R: Ok let us move to the left hand side. What will be the next number after 0.74999

S127: It will be 0.749999

R: The next?

S127: It will be 0.7499999

R: So which number is being approached here?

S127: It is 0.75.

R: Now let us go back to the right hand side now which number is being approached here?

S127: It is 1.9.

R: So in completing the gaps on the symbolism here you would replace 1 with which number and 2 with which number?

S127: With 0.75 and 1.9.

The subject acknowledges that he was thinking of whole numbers only as types of numbers. But when asked to mention the function values that would be obtained as $f(x)$ tend to L, he realises that the values that are being approached are 0.75 and 1.9. While there can be many approximations of the same number, the limit value is the unique value which is not obtained by approximations but through the limiting processes. Two is a number that could be considered close to the function values, but in terms of the uniqueness of the limit values, 1.9 and 0.75, are the only values that could be obtained through the limiting process. Through engaging with the interview questions, the subject conceptualisation of the idea of limit improves.

Approximating or rounding off is the limiting process

Of the 88 subjects who obtained 2 and 1 as the answers, three of them (S48, S88, and S136) had done so by rounding off 1.9 and 0.75, the correct values to the nearest whole numbers. The subjects' responses were written as follows: "$f(x)$ approaches $1.9 \approx 2$" and "x approaches $0.75 \approx 1$". S129, an interviewee, was among the subjects who gave 2 and 1 as responses but had not shown in their work that they had approximated. However, this was reflected in an interview. The discussion with S129 now follows:

R: You say that as x approaches 1, $f(x)$ approaches 2. Let us look at this table of values can you add five more numbers in both the column of x and that of $f(x)$. Which number is x approaching? Which number is $f(x)$ approaching? Study the numbers carefully.

S129: (Writing) 0.749999

R: (Interrupting) How many 9's do you have here?

S129: Four because there they were 3.

R: How many do you think we will have in the next number?

S129: Five.

R: Next.

S129: Six, seven, …

R: Will we ever come to an end of those numbers?

S129: No.

R: So which number do they keep on approaching? Because it looks like you were concentrating on whole numbers only.

S129: I was not aware of other specific numbers.

R: Which number do you think would be close to 0.7499999 ..

S129: It is 0.8.

R: Can you think of another number close to 0.74999999…

S129: It is 0.75.

R: Between 0.8 and 0.75 which one do you think is closer?

S129: It is 0.75.

R: Let us concentrate on $f(x)$. Which number do you think $f(x)$ is approaching?

S129: It is 1.9.

R: Can you now write your new answers in the correct position using symbols in b(i)? If I may ask, how did you come up with 1 and 2 as your answers?

S129: I was thinking of whole numbers only because I approximated.

S129 reveals at the end of the discussion that he obtained the values 2 and 1 by approximating. This is an indication that instead of using the limiting process to obtain the answers, approximation was used. But as in the case of S127, possibly the ambiguity inherent in the phrase 'as close as we please' is still a problem. S129 first mentions 0.8 as the number that is being approached. He only changes to 0.75 when asked if there is no other number he can think of. It could also be that the subjects see the real line full of breaks. Otherwise the numbers they had given as their answers were not the ones being approached through the limiting process.

Number of decimal places determines the limit value

S125 was among the subjects whose response was that "$f(x)$ approaches ∞" and "x approaches ∞". When asked how she came about ∞ as the answer, this is what she had to say:

S125: By counting the number of 9's as the pattern continues. The 9's keep on increasing without coming to an end.

What the subjects might have done here is to concentrate on the nature of the $f(x)$ values and the x values rather than the numbers that they are tending to or approaching. As we keep on considering more x values we seem to be getting to a point where the values may consist of infinite number of 9's. This seems to be a problem of interpretation of the phrases '$f(x)$ approaches' and 'x approaches'.

Epistemological obstacles in translating from numerical to algebraic representation

In responding to questions 5(b)(i) and 5(b)(ii) on translation the following conceptions acted as epistemological obstacles.

- The limit value is equal to the function value (11 subjects);
- The limit value is the function formula (6 subjects); and
- The limit value is the lower or the upper bound (15 subjects).

The limit value is equal to the function value

Some extracts showing the above conception are:

S178: $f(x)$ is equal to 1.9 as x approaches 0.75.

S122: As x approaches 0.75, $f(x)$ is equivalent to 2.

S131: Function as $x \rightarrow 0.75$ is 2.

When one looks at the structure of the symbolism $\lim_{x \to a} f(x) = L$, what seems to connect L and $f(x)$ is the equal sign, '=', this is a structural problem. Hence the subjects say that $f(x)$ is equal to L. The component with the limit symbol and the domain process is left out. Thus the opaque symbolism here acts as an obstacle.

The limit value is the function formula

Unlike in the preceding category where the limit value was taken to be equal to the function value, in this category the formula was used instead. The extracts that follow show this:

S88: $\lim_{x \to 1} f(x) = 2x$

S107: $\lim_{x \to 0.75} f(x) = 2x + \frac{x}{2}$

S230: $\lim_{x \to 1} f(x) = 2x + 0.4$

This conception also appears to be both a structural and familiarity problem. The subjects are possibly used to seeing functions in algebraic mode. So when seeing *f*(*x*) followed by an equal sign, it appears to them that what should follow is the relation that connects the *x* and the *y* values. So, the subjects gave the formula which they would use in substituting *x* values to get *f*(*x*) values. That is, if the given *x* values are substituted in any of the formulae given by the subjects, approximate y values to the ones that appear on the table will be obtained. As pointed out in chapter 4, one of the limitations of the study is carrying out interviews before an in-depth analysis of questionnaires is done. This category of response does not have any interviewees in it.

The limit value is the upper or the lower bound

The extracts of this category now follow:

S63: $\lim_{x \to 2} f(x) = 1.8999$

S148: $\lim_{x \to 0.7} f(x) = 1$

S190: The limit of *f(x)* when *x* approaches 0.7 is 1.

S63 takes 1.8999 to be the limit value because it is the last function value given or it is the upper bound. S148 gives 1 as the limit value because it is the lower bound. For the value that the *x* values tend to, S148 chooses 0.7 which is also the lower bound for the *x* values. S148 is very consistent with the direction of movement in determining the values that are approached by both the *x* and the *y* values. It is not clear how S63 gets 2 as the value that the *x* values are moving towards, not unless the *x* and the *y* values columns have been confused with one another.

Epistemological obstacles in translating from algebraic to descriptive mode

Conceptions that acted as epistemological obstacles are:

- The deep structure and the surface structure follow the same pattern (6 subjects);
- The limit value is a dynamic object (9 subjects); and
- The alternative syntactic surface structures are read the same (19 subjects).

The deep structure and the surface structure follow the same pattern

The symbolic representation of limit $\lim_{x \to \ldots} f(x) = \ldots.$ does not have the same structure as the words expressing its meaning, the limit of *f*(*x*) as *x* tends to *a* is …. The part $\lim_{x \to \ldots}$ appears as if the sentence that goes with the limit should be directly said along with $x \to \ldots$ because they are grouped together. That is, the limit is part of the domain. Further confusion is brought about by the *L* that has to be on the right hand side when we already have the part with 'lim' on the left hand side. In terms of surface structure the symbolism lim $f(x) = \ldots$, $x \to a$, would read better as it shows that the limit of the function *f*(*x*) is a certain value, implied by the equal sign, as *x* approaches *a*. This claim is confirmed by the responses of S39, S59, S80, S85, S172 and S231 which read: "(when) the limit of *x* approaches ….", this statement takes the part, $\lim_{x \to \ldots}$, to correspond to the semantic structure.

S59: As the limit approaches *x*, *f*(*x*) approaches 2.

S231: When the limit of *x* approaches 1 the function is 2.

There were no interviewees also in this category of responses.

The limit is a dynamic object

The extracts that follow give the answers from which the stated conception was inferred:

S50: The limit of *f*(*x*) approaches 2 as *x* goes to 1.

S59: As the limit approaches *x*, *f*(*x*) approaches 2.

S80: As the limit of *x* approaches 0.8, *f*(*x*) approaches 2.

S85: When the limit of *x* approaches zero from the left the function *f*(*x*) approaches.

S102: The limit of *f*(*x*) tends to 1.9 as *x* approaches 0.75.

S172: The limit of *x* approaches 1, the function of *x* approaches 2.

S200: As x approaches 0.7, the limit of $f(x)$ approaches 1.8.

S231: When the limit of x approaches 1 the function is 2.

S247: As x approaches 1 limit of $f(x)$ approaches 2.

Some subjects in this category seemed to read 'lim $f(x) = L$' as 'lim $f(x)$ approaches ...' and '$x \to a$' is also read as the second line on its own. Other subjects seem to read the first block $\lim_{x \to a}$ on its own first as '(when) the limit of x approaches a' and last part '$f(x) = L$' as '$f(x)$ approaches L'. This is a problem that is related to the surface structure of the symbolism used. No interviewees were in this category of responses.

The alternative syntactic surface structures are read the same way

An equivalent form of $f(x) \to L$ as $x \to a$ is $\lim_{x \to a} f(x) = L$ read as: The limit of $f(x)$ as x tends to a is L. Some subjects however wrote:

S14: The more the numbers come closer to 0.75 the more $f(x)$ approaches 1.9.

S25: The function $f(x)$ approaches 1.9 as x approaches 0.75.

Since the equal sign '=' is now used to relate the left hand side with the right hand side we no longer say 'as $f(x)$ tends to L' even though in reality the L has been obtained by observing the behaviour of $f(x)$ values.

Epistemological obstacles in encapsulating processes into objects

For this section the questions were asked in order to create a situation to talk about the limit as an object. The questions tested the students on what they understand the limit concept to be and the applications of the limit concept. Eleven interviewees out of 15 were asked the following open questions:

1. What is your understanding of the limit concept?
2. What do you think the uses of the idea of limit are, either in mathematics or in everyday life?

In responding to the first question, the subjects' responses fell into five main categories namely, the motion or dynamic, the operational, neighbourhood, boundary and the endpoint. The boundary category is a classification similar to that of Williams (1991), even though Williams uses the word bound rather than boundary. However, the two words are used in the same sense. The dynamic category is also a classification used by Williams. It is used on the basis of the process of tending to or approaching. The subjects' responses demonstrating this categorisation are now presented in the next subsection.

The limit as dynamic, operational, neighbourhood, boundary, and endpoint

Four subjects described the limit in terms of the process of approaching or moving towards something, two subjects described the limit in terms of the way operations are performed, the other two in terms of points in the neighbourhood, three in terms of boundary and one as an endpoint. The extracts of these subjects now follow:

The extracts of the motion or dynamic category now follow:

S115: I understand it to the behaviour of a function as it approaches a value.

S118: The limit is the point or a number that the function approaches.

S128: I know that when finding the limit you find the number of a function that goes to infinity. Any number.

S129: … It simply means looking for the function whose limit will be approaching 0 or a particular number.

The extracts of the operational category:

S126: The question is difficult. I do not know much about it because I have met it only in calculations but not in explanations. I know that I can calculate the limit of a function.

S127: I know that something has to be done from a certain value to another number.

The extracts of the neighbourhood category:

S116: When we are looking at a certain value when we want the limit lets say we take 2 we are going to look at the numbers close *to* 2.

S117: When we are looking for the limit we are looking for the number that is next *to* another number.

The extracts of the boundary category:

S122: It is the boundary of something.

S123: It is a restriction we are not supposed to cross.

S124: It is a boundary with a start and finish.

The extract from the endpoint category

S125: It is the endpoint of something.

Subjects in the motion or dynamic category describe the limit in terms of the range process only. There is no mention of the domain process. S115 uses 'it' to show that he refers to the limit concept as an object, a noun. S118 and S128 refer to the limit as a noun, a number or a point. The descriptions provided by S115 and S118 are similar to one of the models of limit held by college students in the study of Williams (1991): "A limit is a number or a point the function goes to but never reaches'. While Williams uses the phrase 'but never reaches', S115 and S118 use the word 'approaches', which may have an implied meaning 'but never reaches'. Among the spontaneous conceptions mentioned by Cornu in chapter 2, we have a limit as a point which one approaches without reaching or a point which one approaches and reaches. In this study the subjects were not explicit on whether the limit is being approached without being reached or as it is approached it is also reached.

In the second category, the operational, the limit of a function is described in terms of performing an action of 'calculating'. The subjects describe the method used in finding the limit and not what the limit is. When the limit value is found by tabular method calculations are done. Numbers in the neighbourhood of a are chosen and substituted in the formula. So, to the subjects this appears to be the method that they use in describing how the limit value is obtained. Perhaps this was a popular method used by this group of subjects in class. Though S127 has been included in this category, there is also an element of boundary in her response. She uses the phrase 'from a certain value to another value' in her response.

The third category, the neighbourhood, the concept is described in terms of choosing points in the neighbourhood of a. The concentration is only on the domain. The idea of neighbourhood is likely to come from the informal definition of the limit. This definition has an element of the process of 'tending to' or the process of 'approaching'. As something is approached it means that one will either be in its neighbourhood or will reach it in the sense of landing on it. In finding limits of functions in a tabular form, numbers in the neighbourhood of $x = a$, are chosen. The only thing lacking here is that the limit value is found by looking at the coordination of the domain and the range processes. S117 clearly refers to the limit as a number, a noun. This means that the limit is perceived as an object.

In the fourth category, boundary, there seems to be some interference with everyday meaning of the word. The boundary is perceived as the limit because it sets the mark beyond which one is not supposed to go. Unlike an endpoint which may be perceived from only one side, a boundary is something that can be set on both sides or around an object. S124 refers to a boundary as something that has a start and a finish. That is, reference is made to the lower and the upper bound. It is likely that a subject with this conception may choose the first and the last numbers of either a sequence or a function as the limit values. For example, it is likely that in a sequence {-1, 1, -1, 1, …}, two limits may be obtained. The lower limit may be taken as -1 and the upper limit as 1. In Frid's (2004) study a subject referred to the limit as a border which one cannot cross. In a study by Davis and Vinner (1986) subjects also referred to the limit as a boundary. S123 also says that we are not supposed to cross this restriction.

The word 'endpoint' in the last category appears on the list of spontaneous models of the word 'limit' provided by Cornu (1991). In Frid's (2004) study subjects also referred to the limit as an endpoint. Davis and Vinner (1986) also obtained the same result. An endpoint refers to a particular point beyond which one cannot go. It is however likely to be reachable. This means that the subjects with this conception are likely to take the last function value as the end point or the limit value. All subjects in this category perceive the limit as a noun, an object. This is indicated by the use of the pronoun 'it'. The everyday meaning of the word ‘limit’ interferes with the mathematical meaning and acts as an epistemological obstacle in describing the limit as an object.

Limit concept: applications in mathematics or everyday life:

S117 and S129 said that they did not know the importance of the idea of limit. This is what the other nine subjects had to say in an interview [the subjects’ previous categories of description are in brackets]:

S115: Studying limits has prepared us to be able to deal with functions. There is one particular case in differentiation where we deal with related rates as well as approximation where you find dimensions that will give you the maximum area of a room, if you master the skill of limits you can work out such problems. [MOTION/DYNAMIC]

S128: It is important in almost everyday life not just in mathematics because we have to have limits. I think there is a limit in everything. [MOTION/DYNAMIC]

S116: It helps in cases where the function is undefined at a certain point and you look at how it behaves as it approaches that point and the limit helps you. [NEIGHBOURHOOD]

S122: We have limits in everything. [BOUNDARY]

S123: Limits are found in everything. There must be limitations to everything. There is an extent to which something has to be done. [BOUNDARY]

S124: Limits deal with many issues. We can have a limit from 0 to 5. [BOUNDARY]

S125: Limits help us to be specific. Let us say we talk about natural numbers, we will be talking about this number to that number. [ENDPOINT]

S126: It has applications because in mathematics it shows that at some point a number may be approaching a bigger positive number or a bigger negative number? And you are able to make conclusions. When I talk about the limit I am thinking of a number. So, it is like you are asking me about the importance of a number. [OPERATIONAL]

S127: When we are doing something we have to be limited as to where to start and where to stop. [OPERATIONAL- BOUNDARY]

The subjects seem to relate the applications of the limit to their understanding of the limit. For example, subjects S122, S123, S124 and S125, from the boundary category, in describing the limit they refer to the application of this concept as setting boundaries to whatever action is taken. S115 seems to be the only better one in terms of stating the applications of limit except that he uses the phrase ‘the skill of limits’ instead of ‘the knowledge of limits’.

The results concerning epistemological obstacles in coming to understand the concept of limit in the context of the function were investigated. The findings reveal the following as the major epistemological obstacles: generalising**;** having encountered situations in which certain conceptions are appropriate, subjects generalise the properties to other contexts in which the conceptions are inappropriate. For example, having met situations in which function values do not exist where the function is not defined, the subjects conceive the limit value to exist only where the function is defined. Having encountered situations in which answers are obtained by computing using only one x value, subjects believe that the limit value can also be computed using one value. Thus subjects do not conceive limit values to be obtained through the limiting processes. The symbols 0 and ∞ are also treated like the other types of numbers. Students are not aware of their uniqueness. The process of approximating is also inappropriately applied to finding the limit values.

The everyday language is also an epistemological obstacle. Having encountered some words in everyday life, e.g., limit, the everyday meaning is now extended to the mathematical context. The syntactic or surface structure of some symbolism is an epistemological obstacle. Subjects confuse this with the semantic or deep structures. Function values were also confused with the limit values. Sequences are special type of functions, it is therefore anticipated that some of these epistemological obstacles experienced in the context of a function will be experienced in the context of a sequence.

As epistemological obstacles in understanding the limit concept were investigated in the two contexts, a function and a sequence, the next chapter gives the results of investigating epistemological obstacles within the context of a sequence.

Limit of a sequence: Results and discussion

The previous chapter has reported on the results of 251 subjects who responded to questionnaire 1 on the limit of a function. This chapter reports on the results of 56 subjects who responded to questionnaire 2 on the limit of a sequence. Eighteen of these subjects were interviewed. The questionnaire composed of questions which required the subjects to find the limit of sequences in different modes of representation, the numeric, the algebraic and the geometrical or graphical.

Subjects in this study will have dealt with a lot of functions defined over an interval from their high school education and will also have drawn their graphs by joining points with a line. Such practices are likely to be generalised to sequences as well. This is because it may seem strange for the students to draw graphs which are not joined by lines. It is also likely that depending on the type of sequences that they have been exposed to in class, they may deny that some sequences are indeed sequences.

Errors committed and problems or difficulties encountered by the subjects in responding to the questions were used as indicators of the existence of the epistemological obstacles. The results have also revealed that even correct answers may be sources of epistemological obstacles. Hence, some parts of the discussions include epistemological obstacles in seemingly correct answers. As in the case of reporting the results on the limit of a function, the three main categories developed from the APOS framework are used in analysing the results on the limit of a sequence.

Epistemological obstacles in interiorising actions into processes

Within the APOS framework, one can arrive at a conclusion about the limit value by a consideration of an infinite number of computations. Some will be performed and infinitely many will not be performed but contemplated. A subject who considers one or two values or a finite number of computations before reaching the conclusion about the limit value shows an action conception. A question that revealed a situation where an action conception was made visible in students' work is:

> **Question 1 (b) Evaluate** $\lim_{n\to\infty} \frac{(-1)^n}{n}$**.**

Part of the schema that was to be applied in responding to the question is:

- **Coordination of the domain process,** $n \to \infty$ **with the range process,** $a_n \to L$; **and**
- **Awareness that** a_n **tends to 0 as** $n \to \infty$. **Hence, 0 is the limit value.**

Within this group, 46 subjects obtained the correct answer 0. Some of the responses given by the individuals were: ∞, -∞, no limit, 1, -1, etc. Of the 46 subjects who obtained the correct answer 0, 16 of them were interviewees. In the questionnaire results, the subjects gave the answer as $\lim_{n\to\infty}\frac{(-1)^n}{n}$ **= 0, and did not show any intermediate steps. The researcher therefore could not tell the procedures that led to the answer. Hence, the assumption made was that the subjects had reached the answer by an appropriate procedure of the limiting process.**

The interviews however revealed that none of the 16 subjects obtained the answer by using the limiting process of either 'tending to' or 'approaching' the limit in the mathematical sense. One subject obtained zero by treating the sequence as a series and used the Ratio Test. The other 15 subjects considered only one number in finding the limit value, thus revealing an inappropriate application of the action conception. On the basis that 15 out of 18 subjects obtained the correct answer by implementing inappropriate procedures, one would suspect that there were still a good number of subjects who were not interviewed who had obtained their answers by the action conception.

During the interviews the three sub-conceptions within the action conception that mainly led to finding the limit value for the 15 subjects were:

- **Approaches means nearer to (4 subjects);**
- Approaches means approximately equal to (6 subjects); and

- Having mixed conceptions, action and process, a transition problem (5 subjects).

The results reflecting these three subcategories are presented next.

6.1.1 Interpretation of 'approaches' as 'nearer to'

In this subcategory the subjects (S1, S17, S19 and S23) reached the conclusion about the limit value by a consideration of only one value, which they considered to be nearer to zero. Hence, the limit value was taken as zero. The extract reflecting this now follows:

Extract of S1:

R: How did you get zero as the answer?

S1: When I thought of n as that big number as the denominator, at the top we will always have 1 or –1. So, when I divide either –1 or 1 by a very big number we will get a number that approaches zero.

In everyday life when one approaches a point, the end result would be finding oneself in the neighbourhood of the point, landing on the point, or surpassing the point. Thus 'approaches zero' in the case of the subjects have been taken as synonymous to 'nearer to'. In the mathematical sense, 'approaches' is a limiting process while 'nearer to' is not.

6.1.2 Interpretation of 'approaches' as 'approximately equal to'

S8, S9, S11, S12, S17 and S21 used 'approximately equal to' as a limiting process. The extracts of S8 individually and S11 and S12 in a group interview are presented next.

Extract of S8

R: How did you get zero as the answer?

S8: I look at the n. If n gets larger that limit will go to zero.

R: What do you mean when you say that the limit will go to zero?

S8: When you take numbers that are large, you divide one negative or positive by a big number, you get approximately zero.

Extract of S11 and S12

R: How did you get zero as the answer S12?

S12: It is by dividing by a very big number.

R: How about you S11?

S11: I see that when n is even I will get one and when n is odd I will get –1 and –1as n goes to ∞ over n will be a very big number and when I divide 1 by a very big number it is approximately zero. Because it is 0.0000…

R: (Intervening) Does it mean that when you divided 1 by a very big number you rounded off?

S11: Yes I rounded off. I approximated it to zero.

In some contexts in mathematics numbers are written to certain degrees of accuracy. Thus 'approximating' is an appropriate operation in those contexts. Depending on the required degree of accuracy different approximations may be found. The limit value is a unique value that is not obtained by approximating but the limiting process of 'tending to' or 'approaches'.

6.1.3 Having mixed conceptions, action and process: transition problem

Five of the sixteen subjects, S4, S13, S14, S24 and S 49 had mixed conceptions, the action and the process conceptions and none of these conceptions was fully developed. The interview extracts of S24 and S49 now follow:

Extract of S24:

R: In 1(b) how did you get zero as the answer?

S24: I know that n is approaching ∞. So when you divide by a very big number you get zero.

R: How many big numbers did you consider?

S24: Many numbers

S49 shows the same transition stage:

R: How did you get zero as the answer?

S49: The numerator is either 1 or –1 and this becomes smaller when we divide by a big number. It gets smaller and smaller.

S49 here at first refers to dividing by a big number, an action, and later says 'it gets smaller and smaller,' the beginning of a process conception. S24 at first refers to dividing by a big number, an action, and later says many numbers, the beginning of a process conception.

This section has presented results reflecting how the action conception was considered as an epistemological obstacle to the understanding of the limit of a sequence. The next section reports on the results showing the type of conceptions that were used inappropriately in an attempt to form a coordinated pair of processes, a process schema.

Epistemological obstacles in constructing a coordinated pair of processes

The coordinated pair of processes considered in this section is $a_n \to L \text{ as } n \to \infty$ via the given sequence. Subjects seemed to have problems in making the stated mental constructions as they made errors originating from a variety of sources of conceptions that have been identified as epistemological obstacles. These include:

- Different modes of representation represent different sequences;
- An alternating sequence is not one but two sequences;
- Points on the graph are joined by a line;
- A well defined sequence has a single formula;
- Approaching or tending to the limit means closer to or nearer to the limit;
- Convergence means meeting at a point;
- The limit is determined by the number of decimal places; and
- The limit value is a dynamic object.

Situations showing errors originating from the stated conceptions are elaborated in the discussions that follow. Since errors originating from the stated conceptions were not mutually exclusive, some examples contain more than one error at a time.

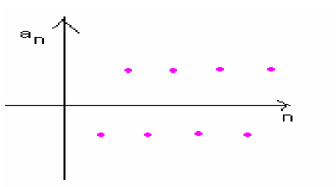
Different modes of representation represent different sequences

Finding the limits of sequences in questions 1(a), 4(b) and 8(b) (see Figure 6.1) is the same process since the same sequence is represented in different modes: an algebraic, a numerical and a geometrical. Finding the limits in questions 1(b) and 8(c) (see Figure 6.2) is also the same process since the same sequence is represented in different modes: algebraic and geometrical. This idea of representation will be revisited in the next chapter when approached from a semiotic perspective. The questions representing the first and the second sequence follow each other. The summary of results for the first and the second sequences are given in Table 6.1 and Table 6.2 respectively.

Question 1 (a) $\lim_{n\to\infty}(-1)^n$ Question 4 (b) The limit of a sequence {-1, 1, -1, 1, -1, …}

Question 8 (b)

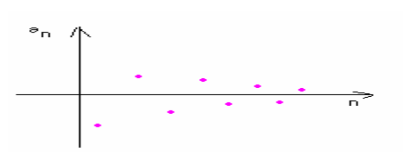
The limit of:



$a_n = (-1)^n$, n = 1,2,3, ...

Figure 6.1 Different representations of the first sequence

Question 1 (b) $\lim_{n\to\infty}\frac{(-1)^n}{n}$ Question 8 (c) The limit of



$a_n = \frac{(-1)^n}{n}$, n = 1,2,3, ...

Figure 6.2 Different representation of the second sequence

Part of the schema that was to be applied in responding to the questions of the first sequence includes:

- Awareness that the representation in each question, represents the same sequence;
- Coordination of the domain process and the range process via the given representation of the sequence to get the limit value; and
- Awareness that the limit does not exist since a_n tend to two values, 1 and -1.

Part to the schema in responding to the second sequence needs the awareness that in each mode of representation a_n tend to 0, as n tends to ∞.

Tables 6.1 and 6.2 give the results of responding to the questions in figures 6.1 and 6.2 respectively. These tables are followed by discussions which show the type of conceptions that led to the identified errors in responding to the two sets of questions.

Numerical and geometrical representations have approximately the same number of subjects. These modes of representation seem to show on the surface only a finite number of points or terms whereas at the deeper level the representations represent an infinite number of points or terms.

Table 6.1 Tabular representation of students' responses for the first sequence

Subjects' responses	Question 1(a) Algebraic representation: Number of subjects	Percentage number of subjects	Question 4(b) Numerical representation: Number of subjects	Percentage number of subjects	Question 8(b) Geometric representation: Number of subjects	Percentage number of subjects
1	8	14	8	14	6	11
-1 when n is odd and 1 when n is even	36	64	24	43	30	53
$\pm\infty$	2	3.6	1	1.8	3	5.4
No limit	0	0	9	16	6	11
∞	6	11	5	8.9	5	8.9
-1	2	3.6	3	5.4		1.8
$-\infty$	1	1.8	1	1.8	1	1.8
0	1	1.8	0	0	2	3.6
Undefined	0	0	0	0	1	1.8
No response	0	0	5	8.9	1	1.8

Table 6.2 Tabular representation of students' responses for the second sequence

Subjects' responses	Question 1 (b) Algebraic representation: Number of subjects	Percentage number of subjects	Question 8(c) geometric representation: Number of subjects	Percentage number of subjects
1	3	5.4	2	3.6
0	46	82	43	77
Undefined	2	3.6	1	1.8
No limit	1	1.8	2	3.6
∞	1	1.8	1	1.8
-1	2	3.6	3	5.4
$-\infty$	1	1.8	1	1.8
$\pm\infty$	0	0	1	1.8
No response	0	0	1	1.8

Finding the limit of the same sequence as the same process required the ability to start with the same object and end up with the same transformed quantity (Dubinsky & Harel, 1992). Though the majority of the subjects get the same answers ±1 in Table 6.1 and 0 in Table 6.2, the numbers were however not uniform across representations. This is an indication that the same sequence in different modes of representations was treated as different sequences. For example, in Table 6.1, 36 subjects get the limit values +1 and -1 for the algebraic representation of the sequence. But only 24 get the same answers for the numerical representation of the same sequence. In both tables, the algebraic and the geometric representations have been treated relatively the same. From Table 6.1, 36 subjects and 30 subjects get the same answers -1 and +1 from the algebraic and geometrical representations respectively. From Table 6.2, 46 subjects and 43 subjects get the same answer 0 also from the algebraic and geometrical representations respectively.

An alternating sequence is not one but two sequences

From Table 6.1 the majority of students get the same answers in finding the limits of the sequence in different modes of representation. The most popular answer is 'the limit is 1 when *n* is even and -1 when *n* is odd'. This answer was obtained by treating the sequence as two sequences instead of one. S13 shared this conception with subjects S11, S15 and S18. He also explicitly said that this was not one but two sequences in an interview.

R: Do you take –1, 1, -1, 1,…to be one sequence or two sequences.

S13: I think these are two sequences.

R: Which are they?

S13: 1, 1, 1, … and –1, -1, -1, …

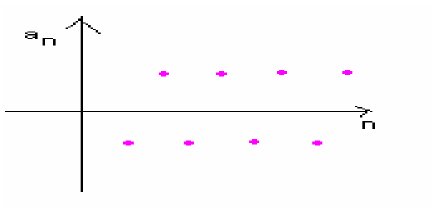
In a study by Tall and Vinner (1981) students regarded a sequence similar to the given one as two sequences. Hence, two limits were obtained for the sequence. The subjects may probably have been exposed to finding limits of either monotonic sequences or constant sequences and transiting to finding the limits of alternating sequences now becomes a problem.

Points of a graph are joined by a line

The question asked was:

Question 8 (b)

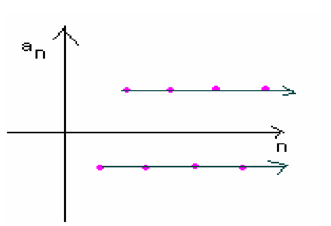
Does the sequence represented by the graph below have a limit? Explain how you obtained your answer.



$a_n = (-1)^n$, n = 1,2,3, ...

Here the subjects were to show that the limit does not exist since the odd terms tend to –1 while even terms tend to 1 as $n \to \infty$. Some subjects found the limit to be ∞ or $\pm\infty$. These answers were obtained by joining the points of the sequence as shown in Figures 6.3 and 6.4. Nine subjects employed this method. S11 was among those who joined the points as in Figure 6.3. In an interview he said that the limit is ∞ because the lines joining the points tend to infinity. S24 and S17 who joined the points as in Figure 6.4 (see next page) said that the limit is $\pm\infty$ because the top line is tending to $+\infty$ whereas the line below is tending to $-\infty$. Alongside the diagram the algebraic representation of the sequence was given as $a_n = (-1)^n$, n = 1,2,3,… and also the same question was asked in both the numerical representation in Question 4 (b) {-1, 1, -1, 1, -1, …} and the algebraic in Question 1 (a). The subjects could not coordinate these modes of representation.

Subjects also seemed to be used to joining points of a graph so much that this was done even in cases where it did not hold. There seemed to be a very serious problem also in conceptualising the domain process. The n values seem not to be considered in order of counting numbers that they correspond to. If so, subjects would have realised that the terms of the sequence are alternating with the same amplitude. Hence, the limit does not exist.



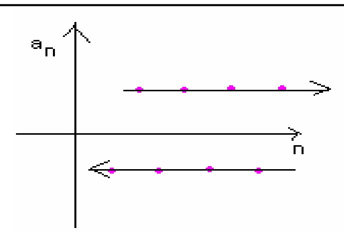


Figure 6.3 The limit as ∞ **Figure 6.4** The limit as ±∞

A well-defined sequence has a single formula

The question asked was:

Question 4(d):	**Find the limit of the sequence: {½, 0, ⅓, 0, ¼, 0,…}**

Only 11 subjects out of 56 subjects obtained the correct limit value zero. Twelve subjects said that the limit of the given sequence does not exist. Eight of these 12 subjects said that the limit does not exist because the terms of the sequence do not tend to any specific value. The remaining four subjects said that the limit does not exist because the sequence does not have the n^{th} term or a formula. Individual responses such as 1, ∞, n+1,…were also given. Twenty subjects, nine of whom were interviewees, did not respond to the question. Some of these 28 subjects made attempts to find the formula, the attempts were however erased as they did not correspond to the numerical sequence. The subjects were not aware that any generalisation made should hold for all the terms, that is, they should obtained the formula that matched with both the odd and the even terms of the sequence:

$$a_n = \begin{cases} \dfrac{1}{(n+3)/2} & \text{for n odd} \\ 0 & \text{for n even} \end{cases}.$$

Davis and Vinner (1986) suggest that problems such as these, are due to the fact that the syllabuses are overloaded with monotonic sequences so the shift to the other types of sequences is very difficult. An interview extract of S19, providing an explanation of why there is a tendency to change to algebraic mode before any work is done by this group of subjects is presented. This is followed by an extract of S11 and S12 who explained why they did not respond to the question.

R: What does it mean to say that the sequence does not have the n^{th} term? Do you mean the formula or what?

S19: I tried to find the formula for the n^{th} term so I could not get it.

R: So you were looking for the formula?

S19: Yes, so I concluded that since I could not find a_n, I could not get the limit.

R: Is it possible to find the limit of the sequence only when you know the formula?

S19: Yes madam, I think it is because I have been doing it like that.

In all the cases, in this group of students, where the sequences were given in a numerical form, the subjects translated these to the algebraic representation and then proceeded with the work to find the limit value. When interviewing S11 and S12 (group interview) who had not responded to the question, this is what they had to say:

R: Both of you have not responded to this question, why did you have a problem with this sequence?

S11: I have never seen a sequence like this.

R: What did you think you were looking for when looking for convergence of this sequence?

S11: Convergence to some particular value.

R: Ok, what was the problem now about that?

S12: I did not find the function that matches the sequence.

R: So which means that you wanted some general formula that would help you.

S12: Yes.

S11 says that he had not seen a sequence like this before. This suggests that he probably did not bother to engage in finding the limit of the given sequence. S12 gives the same reasoning as the other four members of the group that he was looking for the formula for the sequence. For the fact that 28 subjects out of 56 subjects did not respond to the question, this shows that the subjects were either seeing this not as a familiar sequence or they could not find the formula for the sequence as suggested by other members of the class.

Approaching or tending to the limit means being closer or nearer to

The question asked was:

> Question 7: "What is the limit of the given sequence {3.1, 3.14, 3.141, 3.1415, 3.14159, …}? Why do you think so?"

Part of the schema that was to be applied in responding to the question was:

- Coordination of the domain process $n \to \infty$ with $a_n \to L$; and
- Awareness that the given terms of the sequence are increasing approximations of π. Hence, they will approach π as the limit as $n \to \infty$.

Twelve subjects did not respond to the question. Seven subjects got the correct limit value, π. They however, gave incorrect reasoning. All the 7 subjects conceived the given sequence as a constant sequence whose limit value is also a constant. Their responses now follow:

S2: In this case π is a constant.

S4: Because the limit of π is the limit of a constant and hence the result is pi.

S8: Since π is a constant, the limit of a constant number as $n \to \infty$ is a constant.

S18: Since we are just increasing the number of decimal places which just give the value of π.

S38: The limit does not turn to any other specific number.

S51: This is because the limit of a constant number as $n \to \infty$ does not change. That is, it is still the same number.

S56: Because the limit of a constant is a constant only.

Most subjects will have met the number π before their university education in a numerical form written correct to a certain degree of accuracy. So, meeting it again in varying degrees of accuracy all at once, does not register to the subjects that these representations are not the exact values of pi. Hence, they need to be treated as approximations.

Convergence means meeting at the same point

When responding to Question 7, six subjects gave 3 as the limit value. Two subjects gave 3.14 as the answer. Another group of two subjects gave 3.1 as the limit value. The reasons that the subjects gave for their answers now follow:

S1: 3.1, the decimal number that is constant in all values of the sequence is 0.1 and it shows that is does not increase even when approaching the n^{th} decimal place.

S24: Because 3.1 will not change as values are being increased.

S49: This (3.14) is the lowest common limit that is found in almost every case.

S39: The limit here is 3. Since when we increase the number of decimal places of pi the number remains 3 when it is a real number.

S5: Although the number of decimal places are changing the limit approaches 3.

In physics, we talk about the converging rays. These rays converge at the same point. This conception when taken into the context of a sequence could be interpreted as saying all the terms of a sequence converge to the same point. That is they should all have something in common. And to the subjects since that something in common should be possessed by every term then it is to them the digits that do not change. Each term of a sequence is like a ray of light. In the context of a sequence this conception acts as an epistemological obstacle because the terms of a sequence converge only by assuming a certain order, by approaching the limit value in this order.

The limit is determined by the number of decimal places

Ten subjects gave ∞ as the limit value. S19, S41 and S44 were among the subjects who said that the limit of the sequence of the approximations of pi is infinity. The questionnaire responses of S41 and S44 are presented first. These are followed by the interview discussion with S19.

S41: The limit is infinity. The numbers keep increasing.

S44: The limit is infinity. Pi is an irrational number which means the number of its decimal places is infinite.

Extract of S19:

R: You say that the limit of the sequence $a_n = \{3.1, 3.14, 3.141, 3.1415, 3.14159, \ldots\}$ is ∞ as $n \to \infty$. Can you explain how you got this answer?

S19: How I got this?

R: Yes. How did you get infinity?

S19: Madam this one I cannot explain.

R: What do you think 3.1 stands for? What decimal place of pi do you think it is?

S19: The first decimal place.

R: What about 3.14?

S19: The second.

R: As we increase the number of decimal places do you think we are increasing the accuracy or we are decreasing the accuracy?

S19: We are increasing the degree of accuracy.

R: Of what?

S19: Of the number pi.

R: Of the number pi? So as we go on and on, which number are we getting close to?

S19: To 3.

R: Are we getting close to 3? Here the first decimal place is 3.1 and the second is 3.14, are we getting closer to 3?

S19: No.

R: To which number are we getting close to?

S19: 3.2.

R: But here we are told that this numbers are the n^{th} decimal places of pi. That is, 3.1 is pi to 1 decimal place, 3.14 is pi to 2 decimal places, …, the 6^{th} , then as we go on and on you said that we are increasing the degree of accuracy by considering the number of decimal places of which number?

S19: Pi.

R: So as we are increasing the degree of accuracy of pi, which number are we getting close to?

S19: I am not sure about this one.

The subjects in this group considered the number of decimal places of the number π. While S41 and S44 managed to explain how they obtained ∞ as the answer, S19 seemed to have forgotten how he got this answer. Increasing the degree of accuracy of the number pi should have been taken as the process of tending to pi. However, the preceding interview extract of S19 shows that the subject at one point moved to the left to get 3.1 and at another he moved to the right to get 3.2. Though in words the subject was aware that the degree of accuracy of the number pi was increased, it did not occur to him that the limit should be taken as the number pi. It looks like the subject wanted the answer as digits.

The limit value is a dynamic object

In responding to some questions in the questionnaire, some subjects had constructed sentences using the phrase: "… the limit approaches" instead of the order "…approaches the limit." In everyday life one approaches a point by moving towards it. In the mathematical sense, the limit is a static object. So referring to it as if it is something that can be set in motion is a conception that differs from that of the mathematical community. Hence, it acts as an obstacle to the understanding of limits. Some sentences that were constructed by some of the 18 subjects who displayed this conception now follow:

S5: Although the number of decimal places is changing the limit approaches 3.

S9: …the limit will tend to zero.

S13: Because when the limit of any function approaches a specific number, it is said to be finite.

S35: … the limit approaches the fixed number…

S37: It is because the sequence converges if its limit approaches a certain number other than infinity.

S49: The limit is zero when *n* is even. The limit approaches zero when *n* is odd.

S53: The limit of a_n gets closer to 8 as $n \to \infty$.

S55: The sequence converges whenever the limit approaches a finite number.

S56: It is because its limit approaches specific and finite value.

Since the structure of the sentence was disturbed, the meaning also changed. In an interview S49 was asked what he meant by saying "The limit is zero when *n* is even. The limit approaches zero when *n* is odd." This was his response to finding the limit of {½, 0, ⅓, 0, ¼, 0, …}. This is what he had to say:

S49: Here we see this is a constant function (odd terms 0, 0, 0, 0, …). So, the limit is zero and here we have ½, ⅓, ¼, …, the limit approaches zero.

Though this was anticipated to happen from the argument put forward in chapter 2, it was still a surprise to find this type of conception in this group, that is, in finding the limit of a constant function in this case a sequence, there is no motion felt as the number that is said to be approached is already reached.

Epistemological obstacles in encapsulating processes into objects

The question that was asked to check whether or not the subjects, in this case the interviewees, had encapsulated the limiting process into the object was:

When asked to find 'the limit of a sequence', what do you think you are asked to find?

Part of the schema that was to be applied in responding to the questions was:

- Ability for the subjects to refer to the limit of a sequence as a noun.

In responding to the question, 7 subjects reflected errors originating from two types of conception:

- **Everyday meaning of the word limit (5 subjects); and**
- **Exposure to examples of monotonic sequences (2 subjects), that is, sequences that either decrease or increase.**

The limit is the endpoint, interval or boundary

Three subjects described the limit as the endpoint, one as an interval and one as a boundary. The excerpts that follow represent each of the stated categories of responses:

Endpoint category

S9: We are to find the endpoint.

S17: I think by the limit of a sequence we mean where the sequence ends.

S30: We are to find the endpoint of the sequence.

Interval category

S15: I have to find the interval at which the limit exists.

Boundary category

S24: I think I have to find the boundary within which the sequence lies.

Collins Thesaurus (1992: p.423) provides a list of synonyms of the word 'limit'. Among the synonyms the following are mentioned: end, endpoint, and boundary. Cornu's (1991: p. 155) list of spontaneous conceptions of limit that students hold include an interval and the end. Though these conceptions are appropriate in other contexts, they seem to act as epistemological obstacles to the understanding of the limit concept in the mathematical sense. This is because in saying that the limit is the endpoint, one would think that the limit value should be the last term of the sequence. The last term is only identifiable when dealing with finite sequences. The interval and boundary carry with them the meaning that the terms of a sequence within a certain boundary or interval can be said to be limit values. But we know that in any given sequence we can have only one limit value provided the limit exists. In a study by Davis and Vinner (1986: p. 296) some subjects described the limit of a sequence as "… the endpoint for a list of numbers." In Frid (2004) subjects also referred to the limit as an endpoint.

A well defined sequence should either increase or decrease

Here follows the responses from the two subjects in an interview:

Increasing

S18: I am to find if the sequence is increasing.

S19: As that sequence increases it may approach a certain number.

Monotonic sequences are sequences that either increase or decrease. Having encountered these types of sequences, subjects seem to assume that all sequences should either increase or decrease. But in the mathematical context there are other types of sequences such as constant sequences that do not increase and alternating sequences that may not necessarily decrease nor increase. Dealing with examples of the same kind of sequences seems to give subjects problems in dealing with sequences of different kind.

The study has revealed that the limit of a function shares some epistemological obstacles with the limit of a sequence. Such epistemological obstacles include:

- Approximating as a limiting process; some subjects obtained the limit values not by limiting processes such as ‘approaches’ and ‘tending to’ but by approximating. Thus the applicability of approximating was generalised to wrong context;
- Representation; subjects denied that the same sequence in different representations has the same limit. In the context of a function, the interpretation of a piecewise function was a problem. Hence, subjects ended up getting the wrong limit values.
- Infinite/infinity; in cases where the number of decimal places of the function values were repeating or infinite, the subjects said that the limit of functions or sequences in such situations were ∞;
- A limit value is a dynamic object; subjects used phrases like ‘the limit approaches’ to show that they consider the limit value not as a static object but as dynamic; and
- Everyday language; having met some words with certain meanings in every day life such meanings were applied in the mathematical context. Such words include limit and diverge.

While the stated epistemological obstacles were experienced in both the contexts of a function and a sequence, there are some which were experienced in one context but not the other. Surface structure as an epistemological obstacle was experienced in the context of function. In interpreting some symbolism the subjects followed the same order of words as the symbolic structure. Considering the function value as the limit value was also experienced in the context of a function but not a sequence. Having met situations also in which a function value does not exist where the function is not defined, subjects assumed that the limit value also does not exist where the function is not defined. Having met monotonic sequences, subjects thought that every sequence should be monotonic. Since a sequence is a function, it is reasonable for the two concepts to share the same epistemological obstacles.

As mathematical concepts are learned through the use of language and symbolism, in the next chapter the role of language and symbolism in understanding the limit concept is discussed. Epistemological obstacles emerging from the use of language from both the contexts of a function and a sequence are also discussed.

The role of language and symbolism

This chapter discusses the role of language and symbolism in understanding the limit concept as evidenced by this study. The different roles that have been identified from subjects' work are: representation of mathematical concepts, translation between modes of representation, communication of mathematical concepts, and the manipulation of surface or syntactic structures. These roles will be analysed using the semiotic theoretical perspective.

Representation of mathematical objects

Mathematical objects such as functions and sequences can be signified using a variety of representations. These are: the algebraic, the numerical, the graphical, and the descriptive modes. When representations are used the purpose is to draw the attention of the reader not necessarily on the representation but on the signified object. Sometimes different representations may be used to signify the same object. In Questionnaire 2, on the limit of a sequence, in Question 4(b) and Question 8(b), the same sequence was signified using two modes of representation, the numerical and the graphical. The discussion in this section covers these modes of representation.

The numerical representation of the same sequence

The question asked was:

Question 4 (b) Find the limit of the given sequence {-1, 1, -1, 1, -1, …}

In order to respond to this question, it was important first for the subjects to realise that this representation signified an alternating sequence. A diagram modelling this relationship is shown in Figure 7.1.

Figure 7.1 An object-symbol relation

As reflected in the previous chapter, 24 subjects out of 56 subjects regarded the given representation as signifying two sequences. The two sequences were observed as: {1, 1, 1, 1, …} and {-1, -1, -1, …}. As a result of this the subjects obtained two limits for the sequence. An interview extract of S13 is already presented in chapter 6 to confirm this. S3 also gave a similar response to that of S13:

R: So, do you consider this to be one sequence or two sequences?

S3: I consider it to be two sequences.

R: Which one is which?

S3: 1, 1, 1, … and -1, -1, -1, …

There are subjects such as S19 who saw that the representation signified an alternating sequence, though he did not use the term alternating:

R: What kind of sequence is this?

S19: It is an oscillating sequence.

R: You say this sequence is oscillating. Does it have a limit?

S19: No madam I don't think that this one has the limit because I could not deduce the n^{th} term of the sequence. I saw that it is changing from positive to negative.

The subject starts by referring to the given as oscillating. When asked if the sequence has a limit, the subject refers to the difficulty of not finding the formula as the reason for deducing that the sequence does not have a limit. The terms of the given sequence oscillate infinitely between 1 and -1, and it is non-convergent. Hence, it does not have a limit. The terms of the sequence also change signs from positive to negative. It is also an alternating sequence. Towards the end of the discussion, the subject says "I saw that it is changing from positive to negative" to show that the sequence is an alternating sequence. Thus the role played by symbolism here was that of signification.

After recognising that the given representation signified an alternating sequence, the subjects had to build the concept 'limit' by responding to the given question. That is, the question required them to find the limit of the sequence. An epistemological triangle is used to model this. This is accompanied by a discussion on how the subjects attempted to build the concept 'limit'.

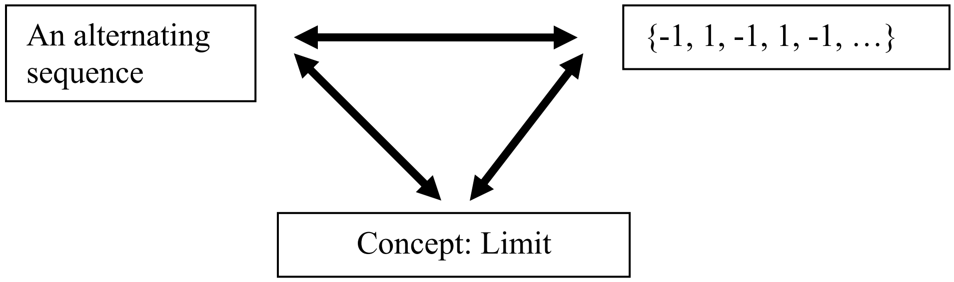


Figure 7.2 An epistemological triangle

The results showing the kind of responses that the subjects gave have already been presented in Table 6.1 of the previous chapter. Here reference will only be made to how the subjects actually responded to the question using language and symbolism. There are three ways in which the subjects responded to the task. These are:

- By using algebraic symbols and the technical language (13 subjects);
- By using algebraic symbols only (39 subjects); and
- By using words or language only (2 subjects).

Two subjects did not respond to the question. The extracts showing responses representing each category are presented next.

Using algebraic symbols and technical language
These extracts show how the subjects in this category presented their work:

S18: $a_n = (-1)^n$

$\therefore \lim_{n\to\infty}(-1)^n =$ -1, if n is odd; 1 if n is even.

$\therefore$ The limit does not exist.

S38: Alternating series. $\lim_{n\to\infty}(-1)^{n+1} = \pm 1$ depending on the value of n (even/odd).

S56: $\lim_{n\to\infty}(-1)^n$ the sequence oscillates.

Though the sequence in this question is represented using numbers, the subjects in this category changed the numerical representation to an equivalent algebraic representation. By doing so, the subjects are aware that using this other mode of representation preserves the sequence. S18 uses the symbol '$\therefore$' in making connections between all the statements as though he is drawing a conclusion at each stage. The intention here was to show that since the sequence can be written as $a_n = (-1)^n$, then $\lim_{n\to\infty}(-1)^n$ = -1 if n is odd and 1 if n is even. Because of the observation that the sequence tends to two values, then a conclusion that the limit does not exist could be drawn. This is indicative in the use of the symbol '$\therefore$' which represents the connective 'therefore'. S38 refers to the given sequence as the series. Two other subjects in this group that referred to the given sequence as a series are S7 and S17. This is not a surprising result "because every series can be understood as a sequence of infinite sums" (Tall & Schwarzenberger, 1978, p. 49). S56 writes the answers in using algebra and technical language such as 'the sequence oscillates'. The subject does not commit whether the sequence oscillates and is convergent or non-convergent. This is the deciding factor for the existence or non-existence of the limit. This subject does not show the work. It seems that all the computations were performed mentally.

Using algebraic symbols only

The subjects in this category also translated the numerical sequence to its algebraic form:

S26: $a_n = (-1)^n$

$\lim_{n\to\infty}(-1)^n = \pm 1$

S36: $a_n = (-1)^n$

$\lim_{n\to\infty}(-1)^n = a_n = (-1)^n = -\infty$

S36 had problems in computing powers of -1. But he still preferred to use the algebraic symbols only in responding to the given task. As shown in the previous chapter, some subjects even said that in cases where the sequences could not be written in the algebraic mode or formula, the limit values could not be obtained.

Using words or language

This is what the subjects had to say about the sequence:

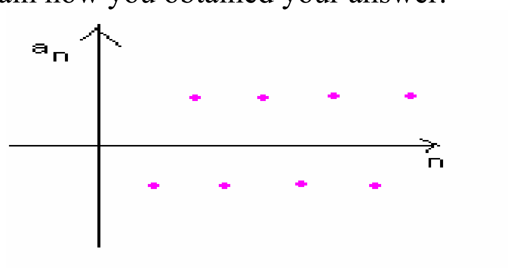
S19: This is an oscillating sequence which is non-convergent.
S35: The sequence does not tend to a fixed limit.

None of the two subjects explicitly says that the limit does not exist. S19 says that the sequence in non-convergent and S35 says that the sequence does not tend to a fixed limit. S19 uses a technical term 'non-convergent'. S36 used an ordinary language 'fixed'. To the subjects the terms used are equivalent to saying that the limit does not exist. Thus in building up the concept limit, the subjects used symbolism, technical language or symbolism and everyday language.

The graphical representation of the same sequence

The question that was asked is:

Question 8 (b): Does the sequence represented by the graph below have a limit? Explain how you obtained your answer.



$a_n = (-1)^n$, n = 1,2,3, ...

The limit is: ……………………. I obtained this limit by: ………………………

In responding to this question, the subjects used the represented mode in a variety of ways. The first part of this subsection investigates how the subjects saw the role served by the graphical representation. Did it signify an alternating sequence? This is an area of concern for the moment.

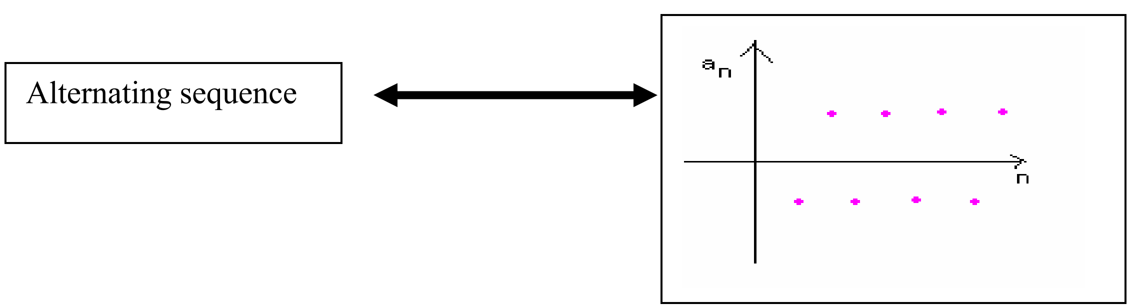


Figure 7.3 An object-symbol relationship

As shown in the previous chapter the majority of subjects perceived the representation as signifying two graphs. This perception was confirmed by their results. Two limit values were obtained. The extract showing this kind of perception in an interview with S17 now follows:

The interview extract of S17 who had $\pm\infty$ as the limit values:

> R: How did you get $\pm\infty$ as your limit values here (pointing at the graph)?
>
> S17: These two graphs (pointing at the top part and the bottom parts of the graph respectively) approaches $+\infty$ because it is going to the right. The bottom one is approaching $-\infty$ because it is going to the left.

The subject sees the graphs not only as two graphs, but he sees them as the graphs of functions by joining points as though the domain of a sequence is defined over an interval. The top part of the graph is said to be approaching $+\infty$ because the points seem to be shifted towards the right. The bottom part is said to be tending towards negative $-\infty$ because the points seem to be shifted towards the left. Thus the assumption made is that these points may continue in either direction as though a sequence is defined at every point on the real line. But we know that the *n* values of a sequence cannot be negative nor can they be any other number besides counting numbers as they signify the position of the terms.

The next part of the discussion looks at how the subjects build up the limit concept using the graphical mode of representation of the same alternating sequence.

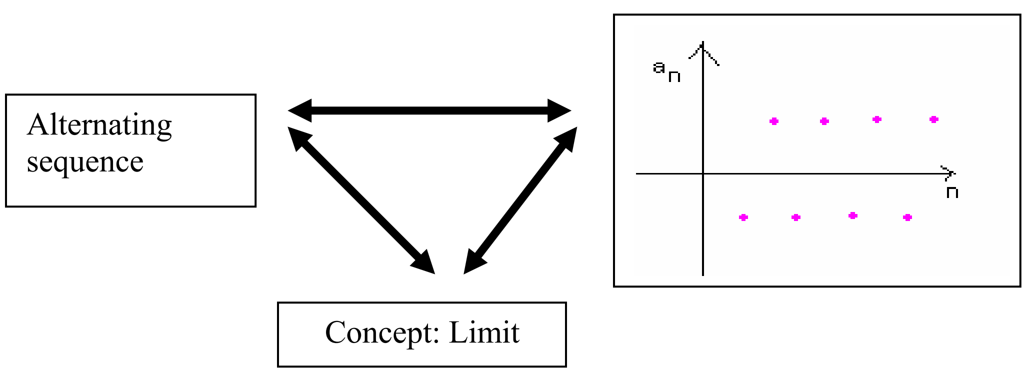


Figure 7.4 An epistemological triangle

The results show that some subjects coordinated the graphical and the algebraic mode, $a_n = (-1)^n$, n = 1,2,3,…, that was provided next to the graph. Other subjects just used the algebraic mode only to obtain the limit values. Thirty nine subjects responded to the question by referring to only the algebraic mode of representation provided next to the graph. Four subjects used the graph only. Eight subjects referred to both the graphical and the algebraic modes of representation. Five subjects did not explain how they obtained their answers. Extracts of students' explanations within these three categories now follow.

Finding the limit value by referring to the algebraic mode only

All the subjects represented here had obtained -1 and 1 as the limit values.

> S12: Looking at the value of a_n. It is always either 1 or -1. It is -1 when n is odd and 1 when n is even.
>
> S33: $\lim_{n\to\infty}(-1)^n$ = {-1 when n is odd.
> = {1 when n is even.
>
> S37: $\lim_{n\to\infty}(-1)^n = \pm 1$

Subjects in this category did not at all refer to the geometric representation. They referred to the algebraic representation provided. In some cases the responses were accompanied by words in explaining subjects' answers. Some subjects gave their answers using the descriptive mode as in the case of S12.

Finding the limit value by using the graphical mode only

The extracts that represent this category of responses now follow. S27 correctly said that there is no limit value. S28 got 0 as the limit value.

> S27: From the graph there is no specific value the limit is approaching.
> S28: Looking at the graph with gradient 0.

Students use representation to build up knowledge in a variety of ways. At a glance S27 sees that the graph is not approaching any specific value. He then concludes that the limit does not exist. The subject however does not follow the proper syntactic structure of the sentence about the limit value. In his explanation he writes: "… the limit is approaching". This conception has been referred to in the previous two chapters, as the dynamic or the motion conception of the limit value. S28 also uses the graph. He takes the top points of the graph as separate from the bottom points. He looks at the gradient of the two pieces whose points he had probably joined mentally and observes it to be 0. S28 does not succeed in building proper conception as he has the problem with the interpretation of the graph and also with relating the behaviour of the graph with the existence of the limit value.

Finding the limit value by coordinating the graphical and the algebraic modes

In building up the concept of limit, the subjects in this category coordinated two modes of representation, the algebraic and the graphical. The extracts that follow demonstrate this. S17 and S53 had obtained ± 1 as the limit values.

> S17: By calculations. But by looking the graph has no limit. This corresponds to the calculations since the limit of an alternating series is finite, then the series is divergent (no limit).
>
> S53: It is clear on the graph that any term does not exceed the two bounds which are ± 1.

S17 is explicit about referring to both the graph and the algebraic calculations. The use of the two modes of representation is implicit in the explanation of S53. The values ± 1 could not be obtained from the graph as the graph did not have any numerical graduations. Hence, the values could only be obtained by referring to the algebraic representation provided. S17 refers to the sequence as the series. As shown earlier on S17 is not the only subject who has used the term 'series' for 'sequence'. From his answer, ± 1, the subject shows in brackets that this means that the limit does not exist. S17 however, has some contradictions in his explanation he says that '… the limit of an alternating series is finite'. Thus one would consider ± 1 finite values. Hence, they are the limit values. Everyday meaning of limit as a bound gets into the way of S53. The subject obtains the limit ± 1, because +1 and -1 are the numbers that cannot be exceeded. They represent both the lower and the upper bounds of the sequence. The subjects in a study by Davis and Vinner (1986), Williams (1991) and Frid (2004) also perceived the limit as a bound.

The discussion has shown that language and symbolism can be used to represent concepts and also in acquiring knowledge. There are some cases in which some representations were interpreted inappropriately by the subjects. In some cases where one sequence was represented, the subjects thought that two sequences were represented. In responding to the questions sometimes the subjects referred to or used representations that they were comfortable to work with.

The next section now looks at how the role of language and symbolism in concept formation was seen as translation between modes of representation.

Translation between modes of representation

The limit concept can be learned in the context of a function and a sequence. As already mentioned, functions can be signified using a variety of representations: the algebraic, the numerical, the geometrical/graphical, and the descriptive modes. Since sequences are special types of functions they also can be represented using the stated modes. Sometimes in solving mathematical tasks learners can be asked to solve tasks which require them to translate from one mode of representation to another. Sometimes the subjects may carry out the translations as a matter of preference in going about a problem. Examples supporting these views have already been discussed in chapters 5. In this chapter, some of these examples will be revisited. The discussion in this section will use questions requiring translation from numerical to the algebraic mode, translation from algebraic to descriptive mode, and translation from the algebraic to numerical mode.

Translating from the numerical to the algebraic mode

In questionnaire 2, in Question 5 (a), a function was represented in a tabular or numerical form as follows (a question already met in chapter 5):

x	$y = f(x)$
0.7	**1**

0.74	**1.8**
0.749	**1.89**
0.7499	**1.899**
0.74999	**1.8999**

The questions on translation were presented as follows:

5 (b) **Complete the expression below so that it is true about the function represented by the table of values above:**

(i) $\lim_{x \to ....} f(x) = ...$

(ii) **After completing the expression above, write in words the meaning of the expression.**

The translation from the numerical to the algebraic form was not a direct translation. This is because first the subjects had to make relations between the values of *x* in the table and identify a number that the *x* values were approaching. In this case the value approached was 0.75. The values of *f*(*x*) in the table also had to be related in order to find the value they were approaching. In this case such a value was 1.9. Substituting the values 0.75 and 1.9 was also not straight forward. In order to perform the right substitution the symbolism $\lim_{x \to ....} f(x) = ...$ had to be connected to the equivalent symbolism of the informal definition of limit, $f(x) \to L$ as $x \to a$. Thus 0.75 would be substituted for *a* and 1.9 for *L.* Knowing where to substitute 1.9 would be a little problematic as the structure of $f(x) \to L$ and $f(x) = ...$ is different. So coordination between the structures had to be made. That is, a subject had to know that the equal sign '=' relates the given function to its limit value as the object.

As shown in chapter 5, only 60 subjects out of 251 subjects managed to perform this translation appropriately. In performing this translation, there are some misconceptions that acted as epistemological obstacles. Some subjects took the limit value to be the function formula. This was done by filling in the formula on the right hand side of the symbolism $\lim_{x \to \ldots} f(x) = \ldots$.. The subjects S88, S107, and S230 wrote:

S88: $\lim_{x \to 1} f(x) = 2x$

S107: $\lim_{x \to 0.75} f(x) = 2x + \frac{x}{2}$

S230: $\lim_{x \to 1} f(x) = 2x + 0.4$

Some subjects also took the limit value to be the function value. Hence, they filled in the function values for the limit values as:

S178: *f*(*x*) is equal to 1.9 as *x* approaches 0.75.

S122: As *x* approaches 0.75, *f(x)* is equivalent to 2.

S131: Function as $x \to 0.75$ is 2.

As already discussed in chapter 5, this could be related to the structural problem. The equal sign '=' between *f*(*x*) and *L* encourage this to happen. The limit values were also taken as the lower or the upper bounds as:

S63: $\lim_{x \to 2} f(x) = 1.8999$

S148: $\lim_{x \to 0.7} f(x) = 1$

1.8999 is given as the last term in the column of function values and 1 is the first term in the same column. Subjects in this case take the limit values as the lower and the upper bounds. Everyday meaning of the word limit again interferes with proper translation.

Translating from the algebraic to the descriptive mode

As already shown in chapter 5, in translating from the algebraic mode to the descriptive mode of representation, some subjects thought that the descriptive representation should follow the pattern of the syntactic structure of the symbolism $\lim_{x \to a} f(x) = L$. For example:

S59: As the limit approaches *x*, *f(x)* approaches 2.

Some subjects took the limit value to be a dynamic object. Hence, the sentences that they constructed read as follows:

S50: The limit of *f*(*x*) approaches 2 as *x* goes to 1.

S59: As the limit approaches *x*, *f*(*x*) approaches 2.

The limit value is a static object. So, taking it to be something that can move is a misconception.

Translating from the algebraic to the numerical mode

Another question in the context of a function that required a translation from one representation to another is Question 6(i). Table 7.1 gives a summary of results in responding to the question. The question is first presented.

Question 6(i): Write five numbers that you would use if you were to find the limit of the function using tables.

$$\lim_{x\to\infty}\frac{\sqrt{x^2+9}-3}{x}$$

Table 7.1 Results of responding to Question 6 (i)

Subjects' responses	Number of subjects	Number of subjects in %
Big positive numbers e.g.S2: 1 000, 20 000, 1 000 000, 10 000 000, 100 000 000	143	57
Big positive and negative numbers e.g. S125: 10 000, 10000 000, 10 000 000, -100 000, -1 000 000.	25	10
Positive whole numbers less than 10 e.g. S95: 1, 2, 3, 4, 5	20	8
Positive or negative numbers in the neighbourhood of 0. e.g. S26: 0.000001, 0.00001, 0.0001, 0.001, 0.1	20	8
Individual responses: No number, 2.2 x 10, and $\frac{1}{0},\frac{2}{0},\frac{3}{0},\frac{4}{0},\frac{5}{0}$	3	1.2
No response	53	21

While 143 subjects out of 251 subjects wrote the correct numbers, 25 subjects wrote numbers that would represent $\pm\infty$; numbers that are either infinitely large and positive or negative. These numbers were taken to be numbers that approach infinity from the left, ∞^{-}, or approach infinity from the right, ∞^{+}, by the subjects. This conception is not acceptable in mathematics. S129, an interviewee, was among the 19 subjects who gave numbers in the neighbourhood of zero. This is how the interview went with regard to this response:

R: You have written 0.1, 0.01, 0.001, 0.0001, and 0.00001 as numbers you would substitute for *x*. Is it positive infinity or negative infinity?

S129: It is positive infinity.

R: When you chose these numbers did you have to choose small numbers or big numbers?

S129: It will be numbers like 999

R: So will you be looking at smaller or bigger numbers?

S129: Bigger numbers.

R: What are the numbers that you have chosen in the neighbourhood of?

S129: They are in the neighbourhood of 0.

The theme of the discussion here is the concept of infinitely large and numbers in the neighbourhood of zero. When the idea of small numbers and big numbers is brought into the discussion, the subject realises that the phrase '*x* tends to ∞' means that the *x* should have been substituted by big numbers. The subject also realises that the numbers that he had chosen are in the neighbourhood of zero. Thus thematic relations were made between the concepts that contributed to the theme of the discussion. The subject seemed to have the concepts of big numbers and small numbers. The problem the subject had was that of coordinating the concept of number with its algebraic representation.

S26 gave the following response to this question:

S26: There are no numbers because infinity is a big number that we don't even know its exact value.

This response shows the conception probably originating from the metaphysical nature of infinity. If a subject cannot relate the concept to the sense perception, then the conclusion made is that infinity does not exist. S136 gives the set of numbers $\{\frac{1}{0},\frac{2}{0},\frac{3}{0},\frac{4}{0},\frac{5}{0}\}$, as numbers she would use in the tabular representation. The subject seems to make a generalisation that any number divided by 0 gives ∞ as the answer. This conception existed in the historical development of the limit concept. Wallis and his contemporaries regarded the concepts of infinitely small and infinitely large as reverse processes. Hence, they made the following generalisation $\frac{a}{0} = \infty$ and $\frac{a}{\infty} = 0$.

Communication

Communication in learning takes place through a variety of ways. Some of these are talking and writing. In mathematics the writing and the talking involve the use of language and symbolism. As mentioned in the theoretical framework chapter each kind of communication has both the thematic and the interactional aspect. The questions used to demonstrate this are from the context of the limit of a sequence.

Within the context of the limit of a sequence some of the questions that needed interpretation and encouraged communication are:

1. What does it mean to say that the sequence diverges?
2. What does it mean to say that the limit is ∞?

The interpretations that were given by the subjects did not match the intended meaning. The following conceptions acted as epistemological obstacles.

- Divergence means tending to infinity; and
- ∞ is a number.

The discussions that follow show how the stated conceptions acted as epistemological obstacles.

Divergence means tending to infinity

Nine subjects associated divergence with the behaviour of rays in lenses. They also made a generalisation that a sequence diverges if it tends to infinity. Even though it is mathematically correct to say that a sequence that turns to infinity diverges, it is not necessarily true that this is the only situation in which we can refer to a sequence to be diverging. Divergence of a sequence is experienced in all the cases in which a sequence does not tend to any one unique value. An alternating sequence such as: {-1, 1, -1, 1, 1, -1, 1,…} diverges because it does not tend to one unique value. The odd terms tend to -1 while the even terms tend to 1. The extracts that follow reflect the conception of divergence that S17 and S9 had:

R: **What does it mean to say that the sequence diverges?**
S17: It is the opposite of convergence. It approaches no number.
R: What does it mean to say that it approaches no number?
S17: It means that it always goes up.

S17 says that the sequence that diverges always goes up. That is, it always increases. This is a misconception as diverging sequences are those sequences whose terms do not converge to a unique value. Thus this subject could deny that an alternating sequence such as {-1, 1, -1, 1, …} diverges since its terms do not go up.
Discussion with S9:

R: What does it mean to say that the sequence diverges?
S9: The sequence diverges when there is no finite answer.
R: Were you meeting the words diverge and converge for the first time in calculus?
S9: Convergence and divergence? I remember the lenses at high school.
R: When responding to these questions, did you use the meaning related to the lenses?
S9: Even though I used the one I learnt in limits even that one I learnt from high school still makes sense because when the light rays diverge they go to infinity and when they converge they meet at a point.

S9 refers to his encounter with the concept of 'divergence' in dealing with lenses in his high school education. He takes the meaning of divergence in that context and uses it here because he believes that the word carries the same meaning regardless of context. The results show that the dual nature of some terminology used in mathematics is a problem.

∞ as the limit

Two subjects had problems with interpreting ∞ as the limit. Their extracts are:

Discussion with S3:

R: What does it mean to say that the limit of a sequence is ∞? Does it mean that the sequence is tending to any specific number?

S3: No I do not think it is tending to any specific number. From my understanding I do not think infinity is an existing number.

Discussion with S23:

R: How did you get -1 and $+1$ as your answers?

S23: When the infinity is odd we will get -1 and when the infinity is even we will get 1.

S3 says that infinity is not an existing number. S23 on the other hand refers to infinity as odd or even, which is rather an unusual situation. However, this conception could be encouraged by the fact that terms such as negative infinity and positive infinity exist. Hence, the subject finds it legitimate to talk about infinity being odd or even.

Manipulation of surface or syntactic structures

When learners are confronted with mathematical tasks, they respond to them through the use of language and symbolism. The choice of whether to use symbolism or language depends on a variety of factors. It could be the way the question is asked, the subjects' choice of method, how the subject sees the role of language or symbolism in responding

to mathematical tasks or how familiar a subject is with certain methods or procedures used in
solving mathematical tasks. The question that is used for the discussion in this section is question 2(iv). The question is now presented.

Question 2(iv): The expression concerning limits is given below.

$$\lim_{x \to 0} \frac{\sqrt{x^2 + 9} - 3}{x^2}$$ Calculate the limit.

In responding to this question there are a variety of routes that the subjects chose. Table 7.3 gives a summary of the methods that were used.

Table 7.3 Methods used in calculating the limit

Method used by the subjects	Number of subjects	Percentage number of subjects	Number of subjects who performed improper manipulations	Percentage number of subjects who performed improper manipulations
Simplification of radicals by rationalising	125	50	62	25
Differentiating using L'Hospital's rule	42	17	36	14
Simplifying $\sqrt{x^2 + 9}$	32	12	32	12
Substituting $x = a$	22	8.8	-	p
Construction of a table	9	3.6	-	-
Computing the left and the right hand limits	3	1.2	-	-
Individual methods	9	3.6	-	-
No responses	11	4.4	-	-

The methods reflected in Table 7.3 differ in degree in terms of the demands for manipulation of the syntactic structures. In this section only those methods which convincingly reflected the role of language as the manipulation of syntactic structures will be discussed. These are: rationalising, differentiation by using L'Hospital's rule, and simplification of the radical $\sqrt{x^2+9}$.

1.1.1 Simplification of the radical by rationalising

In responding to the question, 124 subjects out of 251 subjects started by rationalising. The results on the manipulation of the syntactic or surface structures were categorised as follows:

- Improper simplification (42 subjects); and
- Improper use of symbolism (5 subjects).

Three subjects in this group gave incomplete work. Eleven subjects displayed individual manipulation errors.

Improper simplification

The extracts of S84 and S188 are now used to show how the subjects in this category performed the manipulations:

S84:

1. $\lim\limits_{x \leftarrow 0} \dfrac{\sqrt{x^2+9}-3}{x^2}$

2. $\lim\limits_{x \leftarrow 0} \dfrac{\sqrt{x^2+9}-3}{x^2} \cdot \dfrac{\sqrt{x^2+9}+3}{\sqrt{x^2+9}+3}$

3. $\lim\limits_{x \to 0} \dfrac{x^2+9-3}{x^2(\sqrt{x^2+9}+3)}$

4. $\dfrac{0+9-3}{0}$

5. It's undefined.

In step 2, the product of -3 and +3 is not obtained. The consequence of not carrying out proper manipulation leads the subject to a situation of having the part 9 - 3 in the numerator. This does not allow the simplification of $\frac{x^2}{x^2}$ to be performed. The subject is clearly having problems in multiplying factors that result in difference of two squares or at least she is even not aware that the structure of the numerator is of the form $(a + b)(a - b) = a^2 - b^2$. In step 4 the subject substitutes x by 0. This does not help in eliminating 0 in the denominator. Hence, the subject writes the statement 'it's undefined'. This appears to be a problem of generalising. When division by 0 is experienced in computing function values we say that the function is not defined at this particular point. The sense in which this statement is used here seems to imply that the limit does not exist.

S188:

1. $\lim_{x \leftarrow 0} \frac{\sqrt{x^2+9}-3}{x^2} \cdot \frac{\sqrt{x^2+9}+3}{\sqrt{x^2+9}+3}$
2. $\lim_{x \to 0} \frac{x^2+9-9}{x^2(\sqrt{x^2+9}+3)}$
3. $\lim_{x \to 0} \frac{x^2}{x^2 \cdot 6}$
4. $\frac{0^2}{0^2 \cdot 6}$
5. $\frac{0}{0} = 0$

In step 2, the subject does not perform any simplification. In moving to step 3, the simplification of 9 - 9 to obtain 0 is done. The x inside the radical symbol in the denominator is also substituted by 0. The substitution of zero for only one x in the same expression is not a permissible mathematical operation. The subject now gets an expression in step 3 in which the part $\frac{x^2}{x^2}$ becomes transparent to allow simplification to obtain 1. Instead a further substitution of x by 0 is continued in step 4. This results in getting an indeterminate form of limit which is reduced to 0. Perhaps the subject sees this as the equivalent form of 0 = 0 x 0, unaware that that the product 0 is obtained in every case where multiplication by 0 is performed. But in this case the result $\frac{0}{0}$ meant an indeterminate form of limit. There were 23 subjects out of 62 subjects in this category who did not simplify the part $\frac{x^2}{x^2}$.

Other types of errors that were committed at an individual level that could be mentioned include:

- Multiplying by $\frac{\sqrt{x^2+9}}{\sqrt{x^2+9}}$ instead of $\frac{\sqrt{x^2+9}+3}{\sqrt{x^2+9}+3}$;
- Writing the last part to be simplified as $\frac{1}{\sqrt{x^2}+9+3}$ which yields $\frac{1}{12}$ instead of $\frac{1}{6}$ because the radical sign does not enclose the 9; and
- Simplification of $\frac{1}{3+3}$ given as $\frac{1}{9}$ instead of $\frac{1}{6}$.

In the first case the subject is not aware that multiplying by the form of one, $\frac{\sqrt{x^2+9}}{\sqrt{x^2+9}}$, will not help in the simplification of the radicals. In the second bullet the 9 does not appear under the radical symbol. Hence, it cannot be simplified. This leads to obtaining the result $\frac{1}{12}$. The last type of error is computational. When adding 3 and 3 the subject gets 9. This means that the performed operation mentally by the subject is multiplication.

Improper use of symbolism

In this category subjects did not use the symbols properly. The two symbols that were not used properly are the brackets '( )' and $\lim_{x \leftarrow 0}$. The questionnaire extract of S118 shows the improper use of both symbols. This is later followed by an interview extract with the same subject. Here the subject is asked to explain what he was trying to do in responding to the question the way he did.

S118:

1. $$\lim_{x \leftarrow 0} \frac{\sqrt{x^2+9}-3}{x^2}$$

2. $$\frac{(x^2+9)-9}{x^2\sqrt{x+9}+3}$$

3 $$=\frac{0}{3}$$

In step 2 the subject does not use the brackets to show that x^2 is multiplying the two terms $\sqrt{x+9}$ and 3. The subject also leaves the symbol $\lim_{x \leftarrow 0}$ while the x still exists. The substitution of zero or the process of x tending to 0 to yield $\frac{0}{3}$, is implemented when neither the numerator nor the denominator are in a rational form. Thus, it is inappropriate to say that x is tending to zero when symbolism associated with x is missing.

The discussion with S118 also reflects that the subject had problems in relating the syntactic structures and semantic structures. Since the interview extract to follow is very long, the lines of the extract will be numbered to ease referencing that will be done as part of the discussion.

R: If one gets $\frac{0}{0}$ after direct substitution you say that this is inconclusive, so what can one
do next?
S118: We were taught L'Hospital's rule. In which we will differentiate the top and differentiate
the bottom and then substitute. And if it becomes 0 again you keep on differentiating.
R: Is it what you have done in your solutions? Let us study your solutions? Can you explain
your steps?
S118: I tried to rationalize.
R: What were you doing as you were rationalizing?
S118: I was trying to take the (square) root of $x^2 + 9$.
R: What did you multiply by?
S118: I really have a problem with this. But what I did was to multiply top and bottom

by $\sqrt{x^2+9}$ + 3.

R: Ok let me see if I could help since you say this is your problem. Lets say you have a+b and you multiply it by a-b, what answer do you think you will get?

S118: It gives $a^2 - 2ab + b^2$.

R: Can you look at it again?

S118: It gives $a^2 - b^2$.

R: Now let us consider the top part only. Let us say this is a. You have $\sqrt{x^2+9}$ – 3 you also have $\sqrt{x^2+9}$ + 3. We shall have a - b as the first and a + b as the second. What result did you say you have with a-b times a + b? What result do you think we will have here?

S118: It is $a^2 - b^{2.}$

R: So what is our a here?

S118: It is $\sqrt{x^2+9}$.

R: If we square it what shall we get?

S118: We shall get x^2+9.

R: What is b?

S118: It is 3. So we should get 3 squared which is 9.

R: So what shall we get as our result here?

S118: x^2+9–9.

R: What remains at the top?

S118: x^2

R: What do you have at the bottom?

S118: I have x^2 ($\sqrt{x^2+9}$ + 3)

R: Can you simplify this again?

S118: Yes we will remain with x^2 at the top and at the bottom, which means we shall have

$$\frac{x^2}{x^2(\sqrt{x^2+9})+3}$$

R: In your response to this question in the Questionnaire you substituted 0 at this stage and obtained $\frac{0}{3} = 0$. The x^2 in the denominator seems to be multiplying by $\sqrt{x^2+9}$ but not 3.

Are you aware that the brackets are in the wrong position? Please put the brackets where they are supposed to be. Now is there anything that you can do with the x^2?

41. S118: Yes we will have $\frac{1}{\sqrt{x^2+9}+3}$.

42. R: We now have a rational number at the top. (relating to the process of rationalizing) So,
43. as x tends to zero (domain process) what are we going to have approximately under the
44. square root sign?
45. S118: We will have the square root of 9.
46. R: What is the square root of 9?
47. S118: It is 3 but it can also be –3?
48. R: Are we looking for the positive square root or the negative one?
49. S118: We are looking for the positive square root of 9, which is 3.
50. R: So what will be the limit?
51. S118: It will be one over 3 plus 3, which is 1/6.

S118 had a serious problem with the manipulation of surface structures in rationalising. This is indicative in the words used in lines 7, 9 and 11 respectively: "I tried to rationalise", "I was trying to take the (square) root of $x^2 + 9$." "I really have a problem with this." In lines 7 and 9, the subject uses the word 'try' to show that he did not have confidence and competence to perform the task of rationalising and also of simplifying the radical. Thus he lacked the connection between the syntax and the semantics.

In lines 13 to 17 the subject is asked to expand or multiply out $(a - b)(a + b)$. He gives $a\text{-}2ab+b^2$ which is an incorrect result. When asked to try the multiplication again he obtains the correct result $a^2 - b^2$. This symbolism is related to the given expression in structure. Another indicator which shows that the semantic or deeper structures were not well understood is by substituting zero for x before the expression has neither the numerator nor the denominator in a rational form (line 36). As a result of this in substituting 0 he gets $\frac{0}{3}$ which he leaves as the limit value.

Language issues appear again in line 46. The researcher asks the question "what is the square root of 9?' The subject responds "It is 3 but it can also be –3?" The answer given by the subject is correct in relation to the way the question was asked. The question lacked precision. In real numbers we know that each positive number has two square roots. As in the case of 9, it has two square roots, -3 and +3. But the intended question from the researcher's side was 'What is the principal or positive square root of 9?' The question was however not asked in such a way that it became clear to the subject that we are looking for the positive square root. The lesson learned from this discussion is that when students give answers which are seemingly wrong it may happen that such answers are not so wrong but correct in relation to the way the question was asked. As instructors we tend to judge the correctness of the answer by looking for the answer that matches the intended meaning of the question which may not necessarily match the way the question was asked.

Differentiation by using L'Hospital's rule

Differentiation is one of the methods used to calculate the limit values through the manipulation of the surface structures. It is a convenient method because it deals with functions by following a certain set of rules. These rules involve the manipulation of surface structures. The values obtained through this process have a higher degree of precision. In responding to the given question, some subjects found the limit for both the numerator and the denominator of the expression. They obtained an indeterminate form of limit, $\frac{0}{0}$, which required them to use some alternative methods. The group of students discussed in this section used L'Hospital's rule. In manipulating the surface structures during the implementation of the rule there are some common elements which were found in the manipulated surface structures. These were categorised as follows:

- Problems of simplification (6 subjects)
- Problems with the chain rule (6 subjects)
- Improper use of symbolism (2 subjects); and
- Premature substitution of $x = a$ (2 subjects).

Problems of simplification

The two extracts below represent the type of problems that were encountered in simplification of surface structures by the subjects in this category.

S140:

1. $$\lim_{x\to 0}\frac{\sqrt{x^2+9}-3}{x^2}$$

2. $$\lim_{x\to 0}\frac{\frac{d}{dx}(x^2+9)^{\frac{1}{2}}-\frac{d}{dx}3}{\frac{d}{dx}x^2}$$

3. $$\lim_{x\to 0}\frac{\frac{1}{2}(x^2+9)^{-\frac{1}{2}}\cdot 2x-0}{2x}$$

4. $$\lim_{x\to 0}\frac{x(x^2+9)^{-\frac{1}{2}}}{2x}$$

5. $$\lim_{x\to 0}\frac{(x^3+9x)^{-\frac{1}{2}}}{2x}$$

6. $$\lim_{x\to 0}\frac{\frac{d}{dx}(x^3+9x)^{-\frac{1}{2}}}{\frac{d}{dx}2x}$$

7. $$\lim_{x\to 0}\frac{-\frac{1}{2}(x^3+9x)^{-\frac{3}{2}}\cdot 3x\cdot 9}{2}$$

8. $$\lim_{x\to 0}\frac{13\frac{1}{2}(x^3+9x)^{-\frac{3}{2}}}{2}$$

9. $$\lim_{x\to 0}\frac{13\frac{1}{2}}{2\sqrt{(x^3+9x)^3}}$$

In step 5 the subject distributes the x outside the brackets and gets the expression x^3+9x inside the brackets. The subject experiences generalisation as an epistemological obstacle. The subject has met situations in which multiplication is distributive over addition. In this case the property did not hold since the brackets were raised to an exponent other than one. This situation was brought about by failure to realise that the part of the expression, $\frac{x}{2x}$, in step 4 could be simplified to $\frac{1}{2}$. Realising that when differentiating the top and the bottom of the quotient function the result in an indeterminate state $\frac{0}{0}$, the subject decides to differentiate top and bottom further. The subject ends with an expression which cannot be referred to as the limit value.

S154:

1. $\lim_{x\to 0}\frac{\sqrt{x^2+9}-3}{x^2}$

2. $\lim_{x\to 0}\frac{\frac{d}{dx}\sqrt{x^2+9}-3}{\frac{d}{dx}x^2}$

3. $\lim_{x\to 0}\frac{(x^2+9)^{\frac{1}{2}}-3}{2x}$

4. $\lim_{x\to 0}\frac{x(x^2+9)^{-\frac{1}{2}}}{2x}$

5. $= 0$

Though S154 has been categorised under the subjects who had problems with simplification, the subject also seems to have the problem in applying L'Hospital's rule and use of symbolism. In step 2 the subject does not put $\frac{d}{dx}$ in front of the constant 3 nor does the subject use the brackets around the whole top part of the expression. In step 3, the denominator is the only part which is differentiated. The radical part is only changed to the index notation but not differentiated. The constant is also left as is. In step 4 the subject seems to have used the chain rule and simplification of $\frac{1}{2}$ x $2x$ to get x as the result. The subject does not simplify the part $\frac{x}{2x}$, the consequence of which is to end up with $\frac{0}{0}$ whose result is given as 0. This is an incorrect simplification.

Problems with the chain rule

Two types of manipulative errors occurred in applying the chain rule. The extracts of S5 and S115 reflect this:

S5:

1. $\lim_{x \to 0} \frac{\sqrt{x^2+9}-3}{x^2}$

2. $\lim_{x \to 0} \frac{(x^2+9)^{\frac{1}{2}}-3}{x^2}$

3. $\lim_{x \to 0} \frac{\frac{1}{2}(x^2+9) \cdot 2x - 3}{2x}$

4. $\lim_{x \to 0} \frac{(\frac{x^2}{2}+\frac{9}{2}) \cdot 2x - 3}{2x}$

5. $\lim_{x \to 0} \frac{x^2+9-\frac{3}{x}}{2x}$

6. $= \frac{0+9+0}{0}$

7. $= 0$

In step 3 the subject leaves $-\frac{1}{2}$ resulting from differentiating the outer function. From step 3 onwards the subject does not differentiate the constant 3. The 3 is left out throughout the work. In step 6 the subject substitutes $x = a$ resulting in $\frac{0}{0}$ which is inappropriately reduced to 0. S115, an interviewee, also had a problem with the application of the chain rule. His work is presented. This is followed by an interview extract related to his work.

S115:

1. $\lim_{x \to 0} \frac{\sqrt{x^2+9}-3}{x^2}$

2. $\lim_{x \to 0} \frac{\frac{d}{dx}((x^2+9)^{\frac{1}{2}}-3)}{\frac{d}{dx}(x^2)}$

3. $\lim_{x \to 0} \frac{\frac{1}{2}(x^2+9)^{-\frac{1}{2}}-0}{2x}$

4. $\lim_{x\to 0} \frac{\frac{d}{dx}(\frac{1}{2}(x^2+9)^{-\frac{1}{2}}-0)}{\frac{d}{dx}(2x)}$

5. $\lim \frac{-\frac{1}{4}(x^2+9)^{-\frac{3}{2}}}{2}$

6. $\frac{-1}{4(9^{\frac{3}{2}})} x \frac{1}{2}$

7. $\frac{-1}{8(3^3)} = \frac{-1}{8(27)} = \frac{-1}{216}$

In step 3 the subject does not apply the chain rule properly. He leaves out the derivative of the inner brackets, which is 2*x*. Because of this he is forced to apply L'Hospital's rule for the second time. This is because he has not been able to get the result $\frac{x}{2x}$ which simplifies to $\frac{1}{2}$. This result would enable him to perform the manipulation that would lead to $\frac{1}{6}$, the correct limit value. In step 5, the subject leaves out the symbolism $x \to 0$. He however, continues the manipulation of the surface structures as though this part still exists. This is evident in the disappearance of the x^2 in the next step. The extract that follows is from a group interview of S115 and S116:

R: Which rule is this that your have applied?

S115: It is L'Hospital's rule.

R: OK. So you differentiated the top and the bottom. Let's see how you did that. In differentiating the top here you have $\frac{d}{dx}(x^2 + 9)^{½} = ½(x^2 + 9)^{-½}$. Is there anything that you think you have left out?

S115: I have left out the constant 3.

R: What is the derivative of 3?

S115: It is 0.

R: So it is Ok that you left it out. Anyone of you can respond to the question. My question is, when differentiating $(x^2 + 9)^{½}$ is the answer $½(x^2 + 9)^{-½}$?

S116: (Intervening) He has differentiated only the outer function?

R: **What is the inner one?**

S115: **It is $x^2 + 9$.**

R: **So what will its derivative be?**

S115: **It will be $2x$.**

R: **Is this the part that you forgot?**

S115: **Yes!**

S115 did not treat the given function as a composite function, the result of which was to use the power rule only. Hence, the function inside the brackets was not differentiated. S115 was made aware of this mistake by S116. The key words that were used by S116 in helping S115 are: "He has differentiated only the outer function". The inferential meaning of this statement is that there is an inner function and the brackets acted as a good clue to locating the position of the inner function. Once again there is a communication breakdown between the researcher (the sender) and the subject (the receiver). The question asked by the researcher is "Is there anything that you think you have left out?" The response is "I have left out the constant 3." The answer is correct since the original expression had a 3 in it and in differentiating the subject left the 3 since its derivative is 0. Since the researcher was looking for something else, the derivative of the inner bracket, the subject was probed further to give the answer corresponding to the intended question.

During the discussion the purpose served by the social interaction was to make thematic relations between L'Hospital's rule, the chain rule, the derivative, and the composite function. After this manipulation of structures what did not happen was to ask the subject the meaning of the obtained limit value. Surface structures are manipulated with the purpose to give meaning to the deeper structures and this purpose is often forgotten. This interview confirms this.

Improper use of symbolism

There are areas in which during the manipulation of surface structures some important symbols were left out. An implication of which is that these structures were not related to the deeper or semantic structures. The work reflecting how S127 carried out the manipulations now follows:

S127:

1. $$\lim_{x\to 0}\frac{\sqrt{x^2+9}-3}{x^2}$$

2. $$\lim_{x\to 0}\frac{2x(\frac{1}{2}(x^2+9)^{-\frac{1}{2}}}{2x}$$

3. $$\frac{(x^2+9)^{-\frac{1}{2}}}{2}$$

4 $$\frac{1}{2\sqrt{x^2+9}}$$

5. $$\frac{1}{2x3}$$

6. $$\frac{1}{6}$$

Though S127 gets the correct limit value, there are some important symbols which she left out during the manipulation. In steps 3 and 4, the symbol $\lim_{x\to 0}$ is left out as though the process of tending to 0 has already been implemented. This makes one to wonder where the result of step 5 comes from. The result in step 5 would only be obtained through the limiting process of 'tending to', in this case, 'x tending to 0' so that $f(x)$ would tend to $\frac{1}{2x3}=\frac{1}{6}$. This shows that the manipulation of the surface structures was not connected to the deeper structures. Hence, it did not matter to the subject even if some important part of symbolism was left out as long as the correct answer is obtained. Subjects in the same category had made similar manipulation errors.

Premature substitution of *x* = *a*

Sometimes when students solve problems on differentiation, it is difficult for them to know the stage at which they have to substitute $x = a$. As in the case of the previous discussion correct answers may be obtained while improper manipulations have been performed. In responding to Question 2(iv), this is how S160 manipulated the surface structures:

S160:

1. $\lim_{x\to 0}\dfrac{\sqrt{x^2+9}-3}{x^2}$

2. $\lim_{x\to 0}\dfrac{\frac{d}{dx}(x^2+9)^{\frac{1}{2}}-3}{\frac{d}{dx}x^2}$

3. $\lim_{x\to 0}\dfrac{2x(\frac{1}{2}(x^2+9)^{-\frac{1}{2}}}{2x}$

4. $\lim_{x\to 0}\dfrac{(x^2+9)^{-\frac{1}{2}}}{2}$

5. $\lim_{x\to 0}\dfrac{(0^2+9)^{-\frac{1}{2}}}{2}$

6. $\dfrac{1}{2\sqrt{9}}$

In step 5, the subject substitutes 0 for *x*. This makes the part of the symbolism $\lim_{x\to 0}$ meaningless as there is no *x* at that stage of the manipulation. The subject also leaves the limit value in the form that could be simplified further. $\sqrt{9}$ could be reduced to 3 to obtain the result in the form $\dfrac{1}{6}$.

Other manipulations that were not performed well by individuals in this category were:

- Taking the derivative of $\sqrt{x^2+9}$ as *2x* as though the expression was x^2+9;
- Taking the derivative of $\sqrt{x^2+9}$ as $\sqrt{2x}$;
- Not differentiating the inner brackets of the radical $\sqrt{x^2+9}$, as some subjects did not consider this to be a composite function; and
- Writing $\sqrt{x^2+9}$ in the index notation as $(x^2+9)^2$ instead of $(x^2+9)^{\frac{1}{2}}$.

When referring to the second bullet, this generalisation could be explained by Tall's *generic extension principle*. Since the function to be differentiated has a radical symbol, its derivative should also have a radical symbol. Inability to identify a function as composite has been very common among the subjects in this study. Perhaps they rarely encountered these kind of functions in their learning. The exponent for squaring is confused with that of finding the square root of. These are the inverse operations. Hence, they are probably likely to be confused with each other especially in cases where the symbolism cannot be related to its interpretation.

Simplification of the radical

Subjects started solving the given problem by manipulating the surface structures through the simplification of the radical part $\sqrt{x^2+9}$ and obtaining $x + 3$. That is, all the subjects who started solving the question by simplifying the radical encountered generalising as an epistemological obstacle. While the simplification of $\sqrt{9x^2}$ for $x \geq 0$ would yield $3x$, it is not necessarily the case with the simplification of $\sqrt{x^2+9}$. Four of these subjects (S116, S121, S126, S129) were interviewees. Though the starting point was the same for these subjects, there were some differences in the manipulation procedures towards the end. Seventeen subjects out of 32 subjects resorted to substitution at different stages before the limit value was obtained. After simplifying the radical, six out of 32 subjects, applied L'Hospital's rule before obtaining the limit value. No interviewees were in this group. The extracts that follow show how each group presented their work:

Simplification of the radical followed by substitution

The extracts that follow reflect these manipulative procedures:

S3:

1. $\lim_{x \to 0} \frac{\sqrt{x^2+9}-3}{x^2}$
2. $\lim_{x \to 0} \frac{x+3-3}{x^2}$
3. $\frac{1}{x} = \frac{1}{0}$ So division by zero is not allowed.

S68:

1. $\lim_{x \to 0} \frac{\sqrt{x^2+9}-3}{x^2}$

2. $\lim_{x\to 0}\frac{x+3-3}{x^2}$

3. $\lim_{x\to 0}\frac{0+0}{0^2}$

4. $\lim_{x\to 0} 0$

5. $= 0$

S129:

1. $\lim_{x\to 0}\frac{\sqrt{x^2+9}-3}{x^2}$

2. $\lim_{x\to 0}\frac{(x^2+9)^{\frac{1}{2}}-3}{x^2}$

3. $\lim_{x\to 0}\frac{(x+3-3)}{x^2}$

4 $\lim_{x\to 0}\frac{x}{x^2}(1+\frac{3}{x}-\frac{3}{x})$

5. $\frac{x}{x^2}=\frac{1}{x}=\frac{1}{0}=0$

The first two steps of S3 and S68 are the same. In the third step, S3 writes $\frac{1}{x}$ to show that the simplification of $\frac{x}{x^2}$ is already done. S3 also leaves the symbol $\lim_{x\to 0}$ while S68 retains it. After substituting 0 for **x** in $\frac{1}{x}$ the subject gets $\frac{1}{0}$. After obtaining this result S3 now uses the words or language "So, division by zero is not allowed" to explain why he cannot proceed. In this case generalisation is an epistemological obstacle. A function value would not exist when division by zero is experienced but division by zero in computing limits does not mean that the limit does not exist. Other subjects in this group who had $\frac{1}{0}$ as part of their steps wrote statements like: "is undefined", "does not exist", "the limit is undefined", "the limit does not exist because division by zero is not allowed". All these statements show that the subjects could not relate the surface or the syntactic structures to the semantic or deep structures.

S129, an interviewee, changed the radical notation to the exponential notation before proceeding with his manipulation of the surface structures. The extract that follows shows how the discussion during the interview progressed in relation to his choice of symbolism in the manipulation:

R: **How did you get $x + 3$?**

S129: I did not want to use the square root so I raised $x^2 + 9$ to the power ½. From there I distributed the ½ to get $x + 3$ (That is $\sqrt{x^2+9} = (x^2 + 9)^{½} = (x^2)^{½} + 9^{½} = x + 3$).

R: So you are saying $x^2 + 9$ to the power ½ is equal to x + 3? Is it the same to say $x^2 9$ to the power ½ and $x^2 + 9$ to the power ½? If I square $x + 3$ am I going to get $x^2 + 9$?

S129: No.

R: What is the answer?

S129: It is $x^2 + 6x + 9$.

S129 avoids to use the radical notation. He resorts to the index notation but in the process inappropriate simplification is performed. The subject realises the mistake of this simplification when reference is made to the context in which the simplification works. Raising the product of powers to the power ½ as in the case of $x^2 9$ or $9x^2$ would yield $3x$ or x times 3. But once x^2 and 9 are separated by a plus sign as in the case above the method does not work.

Simplification of the radical followed by L'Hospital's rule

The extracts in this category now follow:

S147:

1. $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+9}-3}{x^2}$
2. $\lim\limits_{x\to 0}\dfrac{(x^2+9)^{\frac{1}{2}}-3}{x^2}$
3. $\lim\limits_{x\to 0}\dfrac{x+3-3}{x^2}$
4. $=\dfrac{1}{2x}$

S192:

1. $\lim_{x \to 0} \frac{\sqrt{x^2 + 9} - 3}{x^2}$

2. $\lim_{x \to 0} \frac{x^{\frac{2}{2}} + 9^{\frac{1}{2}} - 3}{x^2}$

3. $\lim_{x \to 0} \frac{1+0}{2x}$

4. $\lim_{x \to 0} \frac{1}{2x}$

5. $= \frac{1}{2.0.1} = 5$

S202:

1. $\lim_{x \to 0} \frac{\sqrt{x^2 + 9} - 3}{x^2}$

2. $\lim_{x \to 0} \frac{x + 3 - 3}{x^2}$

3. $\frac{d(x+3-3)}{d(x^2)} = \frac{1}{2x}$

Unlike the previous group of subjects, in this category the subjects resorted to L'Hospital's rule before obtaining the limit values. S147 and S202 left the result in the algebraic form, $\frac{1}{2x}$. S192 substitutes the number 0.1 which is in the neighbourhood of 0. In substituting 0.1 for x in $2x$ the subject does not enclose the 0.1 in brackets but it appears that the operation employed in the computation was multiplication. This is implied by the answer 5 obtained through the computation. S202 uses the symbol d at the top and bottom instead of $\frac{d}{dx}$ in each case to show that the derivatives for both the top and bottom are considered. The subject uses the symbols without being aware of the meaning they carry.

This chapter has discussed the role of language and symbolism in understanding the limit concept. Different roles have been identified and the epistemological obstacles inherent in these roles have been identified from the subjects' responses to the given tasks. In making translations between representations, the opaque structure of the symbolism used acted as an epistemological obstacle. In manipulating the surface structures, the subjects focused on the symbols rather than the ideas they represent. Hence, in the process the subjects committed errors that they were not aware of. In communicating, sometimes words with dual meaning or multiple meaning acted as an epistemological obstacle. This is because these words sometimes meant different things to both the researcher and the subjects. The next chapter draws on the important aspects of the study in the form of reflections, conclusions, and recommendations.

Reflections, conclusions and recommendations

In this chapter, the researcher looks back with regard to the experiences encountered and the thoughts that came to mind in pursuing the study. While the investigation concentrated on epistemological obstacles pursuing the study itself was also full of epistemological obstacles for the researcher. The discussion in this chapter will reflect this. The chapter also compares the main findings of the study with those of other research studies conducted elsewhere. The responses to the main questions of the study are given. Lastly possible implications in the form of recommendations are given.

Reflections

The most interesting part of the journey was the realisation that epistemological obstacles are an important part of the knowledge to be acquired and that they are unavoidable. Hence, they should not be associated with lack of intelligence on the sides of students. The nature of the mathematical ideas, the language used, the nature of the teaching, etc. all these in one way or another may act as epistemological obstacles. An interaction with any of these in learning is unavoidable. Thus epistemological obstacles are also unavoidable.

There are some challenges which one met in data collection using interviews and questionnaires. In interviewing the subjects the most challenging task was that of asking questions in such a way that one does not interfere with the subjects' conception(s) that acted as epistemological obstacles. In cases where early in the interview subjects were asked questions that encouraged the overcoming of the epistemological obstacles, at a later stage it was not easy to find the sources of some conceptions related to the committed errors because the subjects had in the mean time achieved a better understanding of the idea in question. This was experienced more at the pilot stage. But, even when great care was taken in the study, there were instances where the conversation was unavoidably driven towards overcoming an obstacle. I believe that though I was a researcher, my role as a teacher interfered with that of conducting research. As a researcher I had to find out the conceptions that were related to the errors that students committed and as a teacher I had to make it a point that such conceptions are overcome. Thus the two roles, a researcher and a teacher, conflicted.

Questions that did not require students to explain their answers were also very difficult to analyse. This is because they did not reveal students' conceptions that gave rise to the given answers. For example, within the context of the limit of a sequence the subjects were asked to evaluate $\lim_{n\to\infty}\frac{(-1)^n}{n}$. The subjects got the correct answer 0, but it was not easy to know that such an answer was not obtained by a limiting process. This was only revealed during the interviews.

Since the study was not developmental, subjects were not observed over a period of time for concept development. This made it very difficult to know in some cases the circumstances under which a particular kind of knowing was acquired. Questions were responded to, based on the knowledge that was already acquired either through lectures, tutorial sessions or by reading prescribed texts. Though the students were not observed over a period of time to see how their conceptions of the idea of limit developed, the researcher's development of knowledge by encountering epistemological obstacles was experienced first hand. To demonstrate how these periods of slow development of knowledge acquisition may come about, I discuss how I personally developed the concept of limit from encountering the term 'epistemological obstacle' and the question: 'Can a function attain its limit?' when reviewing the literature.

Acquiring the concept 'epistemological obstacle'

As a mathematics teacher I have always been interested in teaching the 'slow' learners. Because of this, I decided to pursue a study that will help me understand the problems and difficulties that learners encounter in learning. I then decided to choose the title of my study as: "Obstacles that mathematics students at undergraduate level encounter in understanding the limit concept". As mentioned in the introductory chapter, a lot of undergraduate students encounter problems in learning calculus. In my case this was evidenced by the high failure rate in calculus at NUL. When reviewing the literature I found out that this problem was not unique to the NUL mathematics classroom.

The operational definition that I had suggested for the term 'obstacle' was: An obstacle is anything that hinders the students' progress in learning. I was not aware of how broad the concept 'obstacle' is. With this conception in mind I then handed in the first draft of my proposal to my supervisors. One comment made by my supervisors that took me a very long time to settle was: This is very broad. There are a number of obstacles that may hinder students' progress in learning. Which obstacles do you want to investigate? It was not easy for me to answer this question. When searching the literature I came across the work of Cornu (1991). I read his chapter on limits. Here Cornu introduces a number of obstacles. He mentions the following: the cognitive obstacles, the genetic and psychological obstacles, didactical obstacles and epistemological obstacles. He gives the following explanation to this kind of obstacles:

> The notion of *cognitive obstacle* is interesting to study to help identify difficulties encountered by students in the learning process, and to determine more appropriate strategies for teaching. It is possible to distinguish several different types of obstacle*: genetic and psychological obstacles* which occur as a result of the personal development of the student, *didactical obstacles* which occur because of the nature of the teaching and the teacher, and *epistemological obstacles* which occur because of the nature of the mathematical concepts themselves. (ibid., p. 158).

From reading this passage I was then aware that there are indeed a number of obstacles so I had to choose one type of obstacle as a matter of focus. Because of my passion for wanting to confront the causes of students' problems in learning, I thought that the notion of cognitive obstacle would be the one to investigate. But the idea that the notion of epistemological obstacle is related to the nature of the mathematical concepts never left my mind. A big question that came to mind about the notion of 'epistemological obstacles' was: when we talk about the nature of the mathematical concepts, what actually are we talking about? When doing more reading Cornu has given the following as examples of the epistemological obstacles of the past:

- The failure to link geometry with numbers;
- The notion of infinitely large and infinitely small;
- The metaphysical aspect of the notion of limit; and
- Is the limit attained or not?

Since I was not used to this terminology by then I did not see how it constitutes the nature of the limit concept. In particular I was puzzled by the question 'Is the limit attained or not'? I then continued to review the literature. I came across the work of Sierpinska (1987) where she investigated the notion of epistemological obstacle with humanities students. Here Sierpinska talks about the dual nature of epistemological obstacles. She categorises epistemological obstacles into 'heuristic obstacles' and 'rigouristic obstacles'. Since only a diagram is used in illustrating these obstacles it was very difficult to understand. However, she had referred the reader to her detailed work published in 1985. Accessing this work was a problem as it was written in French as in the case of the work of Bachelard.

In her work Sierpinska has written the four notions which she says are the sources of epistemological obstacles related to limits. These are: scientific knowledge, infinity, function and real number. She used these notions as her framework for the questions she set for humanities students. This information was important in that I compared these concepts to the epistemological obstacles listed by Cornu. What was common between these concepts was that they seem to be the concepts that are encountered when dealing with limits and therefore could be said to constitute the nature of the limit concept. These concepts seem to be unavoidable when learning or discussing the limit concept. I then became confident that I needed to investigate epistemological obstacles. This does not mean that I had a better understanding of epistemological obstacles. I now had to change my topic of study to 'Epistemological obstacles that mathematics students at undergraduate level encounter in understanding the limit concept'. Having made this choice my task was now to clarify the notion of epistemological obstacle in my mind. I came across the work of Sierpinska (1990). Here she writes:

> We know things in a certain way. But the moment we discover there is something wrong with this knowledge (i.e become aware of an epistemological obstacle), we understand something and we start knowing in a new way.… In many cases overcoming an epistemological obstacle and understanding are just two ways of speaking about the same thing. The first is "negative" and the second is "positive" …. Epistemological obstacles look backwards, focusing attention on what was wrong, insufficient, in our way of knowing. (p. 28).

I then struggled with reconciling the interpretation by Cornu and that of Sierpinska. Epistemological obstacles as obstacles related to the nature of the subject matter and epistemological obstacles as a wrong, insufficient way of knowing. And this kind of knowing was said to be negative. I searched some more literature. I came across the work of Brousseau (1997). He gave the following explanations of epistemological obstacles:

> Obstacles of really epistemological origin are those from which one neither can nor should escape, because to their formative role in the knowledge being sought. (ibid., p. 87)

This brought some further confusion. The question that remained for a long time in mind is: How can a wrong way of knowing play an informative role in the knowledge to be acquired? I then came across the work of Hercovics (1989) who used the work of Bachelard as his base for the explanations he gave. I had met authors who had referred to Bachelard as the founder of the term epistemological obstacle. These include: Cornu, Sierpinska and Brousseau. Cornu and Brousseau had even gone further to give quotations from the work of Bacherlard. I did not pay much attention to these quotes as I was really not sure whose translations they were. Hercovics however became very clear that he did the translation himself. The common translation given among the work of these authors was:

> When one looks for the psychological conditions of scientific progress, one is soon convinced that it is in terms of obstacles that the problem of scientific knowledge must be raised. The question here is not that of considering external obstacles, such as the complexity and transience of phenomena, or to incriminate the weakness of the senses and of human spirit; it is in the very act of knowing, intimately, that sluggishness and confusion occur by the kind of functional

> necessity. It is there that we will point out causes of stagnation and even regression; it is there that we will reveal causes of inertia which we will call epistemological obstacles. (Hercovics, 1989, p.61, a translation by Hercovics).

Herscovics ended this quote by writing “translation by the author of this paper”. I felt at ease in using this translation because I could refer to its source. In this quote Bachelard talks about epistemological obstacles as the causes of stagnation in the knowledge to be acquired. He also says that this stagnation occurs as a functional necessity. To me this seemed to be a positive way of looking at an epistemological obstacle. I further looked at the examples of epistemological obstacles that Herscovics has found from the work of Bachelard. These were:

- The tendency to rely on deceptive intuitive experiences;
- The tendency to generalize; and
- The obstacles caused by natural language. (ibid.).

By looking closely at these epistemological obstacles, I found them to be unavoidable situations also. For example, we use natural language in everyday life and mathematics shares some of its technical terms with it. So, when we get into the classroom situation, we cannot in anyway avoid retrieving some of this knowledge, though now in a technical context. Also before getting into higher education where logical deductions are used as methods of proof, we still use our intuitions. Sometimes they are right, sometimes they are wrong but since they are among our available sources of knowledge we do rely on them to a certain extent. I was convinced through this reflective activity that epistemological obstacles are unavoidable. Reading the historical development of the limit concept also confirmed this idea of functional necessity. I therefore started looking at these obstacles as positive because they are the stepping stones to the knowledge to be acquired.

What remained a problem for me from the work of Hercovics is that he associates the term ‘epistemological obstacle’ with the past and he prefers to use the term ‘cognitive obstacle’ at present. This is because he refers to epistemological obstacles as obstacles that were encountered in the development of the scientific knowledge and cognitive obstacles as the obstacles related to individual learning. This perception created a lot of mental conflict. My problem was 'if these epistemological obstacles are the causes of stagnation in the knowledge to be acquired, does it matter by whom and when?' I referred back to the work of Cornu, Sierpinska and Brousseau. These authors show that these obstacles appear only in part in the history, they are also found in educational practice today. Because this view resonated with my understanding of knowledge acquisition, I settled for it.

As highlighted in the beginning of the discussion, the question 'Can a function attain its limit?' was very problematic to me as I did not know what it meant. In the next subsection I present the path I took in coming to understand this idea.

Acquiring the concept 'can a function attain its limit?'

This question appeared in the history of the limit concept and was initially asked as: "Can a variable attain the limit value?" As I had studied calculus through the interaction of mathematics text books only, the question was experienced for the first time in reviewing the literature. The ideas that came to mind during this encounter were: What does it mean to attain the limit value? Is it the same as saying "Will the function values equal the limit value?" "Does it mean the same thing as reach the limit value?" The researcher first referred to the work of Taback (1975). In the work of Taback the question asked in relation to the word 'reach' was:

> A rabbit starts at one endpoint, say A, of a line segment AB. On his first hop, the rabbit jumps halfway from A to B. On his second hop, the rabbit jumps halfway from where he is toward point B. The rabbit continues to hop, following the same rule of correspondence: every time he takes a hop, he jumps halfway from wherever he is toward point B. Does the rabbit reach point B? (p. 111).

In reacting to the stated problem, this is what Taback had to say:

> The answer to the question depends upon the interpretation of the word "reach". The rabbit will certainly not reach Point B, in the sense of landing on B, after a finite number of hops. The mathematician however, says that the rabbit will reach point B, meaning that the rabbit's hops converge to B as a limit point; that is, the rabbit can get and remain within any given neighbourhood around B. (ibid.).

As there was no other explanation encountered by the researcher besides this, the conception of ‘reach’ as being in the neighbourhood of a point stayed with the researcher for a long period of time. Any reflection on this meaning was a comparison with the everyday meaning of the word. In some cases we say we have reached our destinations when we have landed on them. Sometimes we say we have reached some points when we are in their neighbourhood. But this did not in any way answer the researcher’s question on whether or not ‘reach’ and ‘attain’ are synonymous.

In reading the work of Tall (1991a) related to the generic extension principle, Tall explicitly talks about whether or not the limit can be attained. An example given being that the convergent sequence $\frac{1}{n}$ tends to the limit zero, but the terms never actually equal zero. Thus the terms of the sequence cannot attain the limit value. And such an observation was made by looking at the terms of the sequence through the limiting process of ‘tending to’.

With this conception in mind, a mental conflict was experienced at a later stage when reading the work of Juter (2003b). This occurred for two reasons:

1. The equality of the function value and the limit value was not obtained through the limiting process of ‘tending to’ but by considering any function value that was equal to the limit value; and
2. The words ‘reach’ and ‘attain’ were used synonymously. An interpretation that is different from that of Taback.

In Juter (2003b) the subjects were asked whether or not the function could attain the limit value in $\lim_{x\to\infty}\frac{x^5}{2^x}$. One of the responses from the subjects that was considered to be right was, “Yes, for x=0 → $f(0) = \frac{0}{1} = 0$” (p. 86). This example was different from that of Tall. The function here was said to attain the limit value without the application of the limiting process of ‘tending to’ but by considering a case in which any function value would equal to the limit value regardless of position. Here the domain process is, $x \to \infty$, something very far from choosing 0 in the substitution. But even if the domain process was constituted by the symbolism $x \to 0$, substituting $x = 0$ would yield 0 as the limit value. No function value would ever equal to zero as only numbers in the neighbourhood of zero would be considered.

With regard to the interpretation of the two terms 'reach' and 'attain', this is what Juter (2003a, p. 42) has written:

> The question whether limits are attainable seems to be confusing (Cornu, 1991; Williams, 1991). All the students' responses to questions and tasks about it presented in this paper are incoherent. This study shows that the students interpret the definition as stating that the limits cannot be reached by the function. When they solve problems on the other hand, they can see that sometimes limits are attainable for functions.

The problem was still with the interpretation of the term 'reach' from the work of Juter and Taback. Tall had not committed in using the word 'reach' but 'attain'. Another confusion that was not resolved was as to how the function can or cannot attain the limit value. Is it by a consideration of a limiting process as in the case of Tall? Is it by just an observation of any function value that is equal to the limit value? If 'reach' probably means that something is moving towards something, does attain have the same meaning?

Because of the dual nature of the terms that are used in mathematics, it has been advised that such words be discussed in mathematics classrooms so that students may be aware of the meanings they carry in varying contexts. While this suggestion may seem sound to the researcher, at the back of one's mind, one anticipates that some lecturers may have a feeling that in doing so, they will be mimicking some English language lessons by concentrating on the meanings of words. Hence, students may loose interest in mathematics lessons. But how can communication take place when both the sender and the hearer are not sure as to which meaning of the word the other side carries? Discussing the meanings of these words in a mathematics classroom is likely to show the seriousness of difference in meaning of words used in technical and other contexts.

Conclusions

As mentioned in the introductory chapter, this study was conducted in a different context from those of other research studies of similar nature. Some of the studies mentioned in the study were conducted in a computer assisted environment. Some were conducted in places where the native language was used as a medium of instruction. And some were conducted in cases where the researchers themselves were involved in the teaching of the subject matter. This was not the case with the current study. The current study was conducted in a non-computer assisted environment. The concept development was also not observed over a period of time. Some indicators of how the limit concept might have developed or was developing in the mind of the subjects, were observed in interviewing the subjects. Besides these differences there are however, some similarities in the findings regardless of context. The discussion in this chapter will identify both the differences and the similarities of the findings.

Within the contexts of both a function and a sequence over-generalising was an epistemological obstacle. The limit values were said to exist only where a function was defined. While the function value does not exist where the function is not defined, this is not true with regard to the existence of the limit value. The limit may exist where the function is not defined. Approximating was inappropriately used as a limiting process. Subjects substituted only one or two x values in the computation, rounded off the result, and gave the approximated value as the limit value. Whereas different values can be obtained in approximating the same number depending on the required degree of accuracy, the limit value is a unique value that can be obtained through the limiting process of either ‘tending to’ or ‘approaching’ only.

Having been exposed to examples of monotonic sequences, some subjects assumed that all sequences must be monotonic. In some cases some sequences such as the alternating sequences were treated not as one but two sequences. A study of Tall and Vinner (1981) reflected the same results. Subjects assumed that all sequences should be monotonic. They also considered some alternating sequences to be two sequences. Subjects also assumed that the power rule is applicable to all functions. Hence, it was applied to composite functions as well instead of the chain rule. Similar results were obtained in the study by Clark *et al.* (1997). In the study of Clark *et al*, the subjects did not apply the chain rule to composite functions. The subjects in this study also could not identify the inner brackets of the composite function.

Everyday language also acted as an epistemological obstacle. The limit was interpreted either as an endpoint, a boundary, or a restriction. While these interpretations are true in everyday life, they are not necessarily true in the context of the limit concept. The limiting process of 'approaches' was taken to mean either 'nearer to' or 'approximately equal to'. Hence, the limit values were found by choosing numbers that were nearer to the computed function values. In some cases the limit values were found by rounding off the computed function values. Convergence of a sequence was also interpreted as meeting at one point. This was associated with the meaning used in dealing with the rays of light 'meeting at the same point'. The terms of a sequence were said to meet at the digit(s) that seemed not to change. For example, in considering the sequence 3.1, 3.14, 3.141, 3.1415, … the number 3.1 was taken as the limit value because it consists of digits that are the same across the terms of the sequence. So, the terms of the sequence are said to converge to this number as the limit value.

The role of language and symbolism was identified as: representation of mathematical objects or ideas, translation between one representation and another, manipulation of surface or syntactic symbolic structure, and communicating ideas in the form of writing or talking. In representation while some subjects were aware of the signified by some symbolism, some subjects did not. For example, an alternating sequence $a_n = (-1)^n$ was represented numerically, algebraically, and graphically. The majority of the subjects perceived these representations as signifying two sequences instead of one. Because of this, they obtained two limit values. Some subjects treated the graphical representation of the sequence as though it was a graph of a function defined over some interval. Hence, they joined the points by a line. The subjects were not aware that the graph of a sequence is made up of discrete points because the members of its domain are counting numbers.

In translating between modes of representation, the opaque structure of some mathematical symbolism was also found to be an epistemological obstacle. Subjects seemed to follow the syntactic structure of the symbolic representation in translating from the algebraic to the verbal or the descriptive mode of representation, e.g., in translating the symbolism $\lim_{x \to a} f(x) = L$, to the verbal or the descriptive mode, the subjects constructed sentences such as '… the limit as *x* approaches'. That is, the phrase follows the same structure as the part $\lim_{x \to a}$. There are some subjects who also wrote some formulae in the place of *L* because of the structure, e.g., $\lim_{x \to a} f(x) = 2x + 0.4$. The subjects seemed to have concentrated on the part, $f(x) = ...$ This is probably because they are used to situations where this symbolism is used in representing functions algebraically. Orton (1983a) also found out that the structure of the symbolism in understanding the limit concept is an epistemological obstacle. However, his example was concerned with the different interpretation of the symbolism with the same syntactic structure e.g. 3*x* and *dx*. The symbolism 3*x* means 3 times *x* but *dx* does not mean *d* times *x*. The *dx* is the differential that has to be observed as one.

In communicating mathematical ideas, in some cases the same words carried different meanings between the researcher (sender) and the subjects (receiver). For example, when the subjects were asked what it means to say that a sequence diverges, one of the interpretations given was that divergence means tending to infinity. So, over-generalisation here acted as an epistemological obstacle. Though a sequence that tend to infinity diverges, this is not the only case of divergence that exists and therefore cannot be generalised. Everyday language therefore acted as an epistemological obstacle in communicating.

The manipulation of the surface structures was done instrumentally by the subjects. For example, when finding the limit of the quotient function $\frac{\sqrt{x^2+9}-3}{x^2}$ as $x \to 0$, rationalizing and L'Hospital's Rule were used because finding the limit value for both the numerator and the denominator produced an indeterminate form of limit $\frac{0}{0}$. In the process of implementing these methods, the subjects obtained $\frac{x}{2x}$ or $\frac{x^2}{x^2}$ as part of the expressions that were manipulated. These were not simplified to $\frac{1}{2}$ or 1. Instead the substitution of 0 was done at this stage. The consequence of this was to obtain $\frac{0}{0}$. Instead of perceiving $\frac{0}{0}$ as an indeterminate form of limit, this part was simplified as $\frac{0}{0} = 0$. The subjects did not even realise that division by zero is meaningless. They also did not realise that the process of rationalising was incomplete at this stage as the numerator in this case was not yet in a rational form.

Over-generalising was also a problem. The radical part was simplified as $\sqrt{x^2+9} = x+3$. The schema for simplifying expressions such as $\sqrt{9x^2} = 3x$ for $x \geq 0$ was applied to the wrong context. The symbolism $\lim\limits_{x \to 0}$, was also used inappropriately. It was left out as part of the expression even before substitution of 0 was done. Thus the subjects did not attach any meaning to it.

There are certain recommendations that can be made, implied by the preceding discussions. They are presented in the next section.

Recommendations

From the preceding discussions, the following recommendations are made:

- **That students be exposed to different kinds of representation of the limit concept using simple functions at the beginning because of lack of computer technology and graphical calculators. The possibility of using graphical calculators or computers will allow students to handle complicated functions or sequences. For example finding limits of functions such as $\frac{\sqrt{x^2+9}-3}{x}$ as x tends to ∞, may need availability of graphical calculators. This is because in finding limits of such functions students tend to assume that the limit value is 0 because the limit of the component function $\frac{1}{x}$ as x tends to ∞ is 0.**
- **That students be given opportunity to translate mathematical objects from one form of representation to another and find limit values in these different forms. The algebraic mode seemed to be the most dominant with the subjects;**
- **That students be aided to construct both the domain and the range processes not only by using the symbolism $x \to a$ and $f(x) \to L$. The phrase 'tends to' or the term 'approaches' should be explained in relation to the symbolism. Students seem to think that the process of 'tending to' can be achieved by approximating which is a consideration of a finite number of computations. They have to be aware that the limiting process involves an infinite number of computations.**
- **Attention should also be paid to looking at the idea of limit as a coordinated pair of processes so that the domain and the range processes are looked at simultaneously in finding limits.**
- **Students should be made aware that the function values and the limit values are different mathematical objects.**
- **The students should also realise through a number of examples that the existence of limit of a function does not depend on the function being defined at a point.**
- **Students should also be exposed to a variety of examples of sequences so that they do not encounter problems in dealing with sequences of different kinds.**

- **Words with dual meaning or multiple meaning should be discussed in mathematics classrooms in order to alert students that they are now in a technical context which requires meanings that suit its register.**
- **Students should be aware that the meanings that they bring into the mathematics classrooms are not necessarily correct in every situation. So contexts in which the meanings are inapplicable should explicitly be discussed with students.**
- **In performing processes such as differentiation and rationalising, for example, emphasis should not only be put on proper manipulation. Answers obtained should be explained in terms of the deeper structures.**
- **Lecturers should be aware of epistemological obstacles that students are likely to encounter by referring to both the literature and by engaging critically with the recommended calculus text books;**
- **In investigating epistemological obstacles future research should not only concentrate on the wrong answers, even the correct answers should be investigated. Didactical situations that would promote the overcoming of these obstacles should also be investigated. This is because overcoming these obstacles and understanding are complementary processes (Sierpinska, 1990).**

Studies that could be pursued in the future in responding to some of the raised concerns are:

1. **How can mathematics students at undergraduate level overcome epistemological obstacles in coming to understand the limit concept?**
2. **What kind of epistemological obstacles are students likely to encounter in reading calculus textbooks?**
3. **To what extent does calculus teaching encourage learning that relates the surface structures to the deeper structures?**

While the questions raised could address some concerns arising from the conducted study, conducting the current study also has been useful in a variety of ways. The study has addressed some concerns raised at the beginning of the chapter. That is, it has highlighted some epistemological obstacles that students at undergraduate level are likely to encounter in learning the limit concept. This is a contribution to the existing mathematics education literature. The researcher has also gained some knowledge of the nature of the limit concept. The study has also made the researcher appreciate more the mistakes that students make as they are the rich sources of the causes of stagnation in the learning of mathematics concepts. It is likely that the readers of this report may also develop the same attitude.

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Appendices

Appendix A

National University of Lesotho

Pre-Entry Science Programme

Mathematics for Common First Year Science

May-July 2002

Aims

1. To bridge the gap between the high school and university content.

2. To upgrade the high school mathematics content.

3. To help students transit into higher mathematical thinking

Content

1. **Real Numbers**

The real number line

Sets of real numbers: Natural or counting, whole numbers, integers, rational and irrational

Properties of: Commutative, Associative, Identity, Closure, Distributive, Inverse, Multiplication property of zero, Zero product property.

Equality: Reflexive, symmetric, transitive, substitution

Inequality: Trichotomy, transitive

2. **Algebraic Expressions**

Definition

Polynomials: Identification of monomials, binomials and trinomials.
Finding the degree of, addition and subtraction of.

Expansion of binomials using Pascal's triangle

Finding the specified term of the expansion of a binomial

Factoring: factoring out common factors, factoring by grouping, difference of two squares, sum and difference of two cubes.

Test for factorability: Relating b and c in the expression of the form ax^2+bx+c, $a=1$.

Rational expressions: addition, subtraction, multiplication and division

Simplification of algebraic expressions (in general)

3. Base, Exponents and Radicals

Positive exponents, negative exponents, zero exponent, rational exponent

Properties: Product rule, quotient rule, power rules.

Simplification of expressions with exponents

Writing numbers using the scientific notation

Using scientific notation in making calculations/computations

Using radical notation, simplification of radicals

4. Functions

Definition

Modes of representation: Algebraic, Graphical/geometrical, Numerical, Verbal or Descriptive.

Domain and range

Sketching of graphs

Linear: gradient, y-intercept, point-slope equation of a line, slope-intercept equation of a line

Quadratic: standard form, vertex, maximum and minimum values.

Rational: factoring, intercepts, asymptotes.

Composition of functions

Inverse functions, the process of finding inverse functions

Tests: Vertical and Horizontal line tests

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Appendix B

M111-5 Algebra, Trigonometry and Analytic Geometry 2001/2

Syllabus

Set notation. Permutations, combinations and binomial theorem. Real number system. Radicals, inequalities and absolute value. Partial fractions. Coordinate Geometry: equation of a straight line and circle. Introduction to conics. Relations, functions and their rational zeros. Mathematical induction. Introduction to the arithmetic of complex numbers. Powers. De Moivre’s theorem. Rational functions, inverse functions, exponential and logarithmic functions. The circular functions, identities and their graphs. Arithmetic and geometric series.

Course outline

Algebra

1. Set notation – Logic and sets

2. Permutations and combinations.

3. Operations with real numbers: The real number, elimination, inequalities, the remainder and factor theorems, rational functions, partial fractions, indices, inverse functions, logarithmic and exponential functions, equation in which the unknown is an index.

4. Finite sequences and series: Sequences and series, the finite geometric sequences and series, the infinite geometric series.

5. The binomial theorem: The binomial theorem for positive integral index, proof of the binomial theorem when n is a positive integer, mathematical induction.

6. Complex numbers. Definition and examples of complex numbers, the rules for manipulating complex numbers, geometric representation of complex numbers, cube roots of unity.

7. The quadratic function and the quadratic equations: The general quadratic equation, the quadratic function, the relation between the roots of a quadratic equation and the coefficients.

Trigonometry

Angles in radians and degrees, trigonometric ratios for an acute angle and the any angle. The graphs of the trigonometric functions, addition formulae, De Moivre's theorem and exponential functions, multiple and sub-multiple angle formula, inverse trigonometric equations involving different ratios of the same angle. Solution of triangles – the sine formula and the cosine formula.

Analytic Geometry

Cartesian coordinates, polar coordinates, the transformation of one into the other. Distance between two points in terms of their Cartesian coordinates, co-ordinates of a point which divides the line joining two given points internally and externally in a given ratio. The area of a triangle in terms of the co-ordinates of its vertices, condition for three points to be collinear. The straight line, the circle, the ellipse the hyperbola- equations and points of intersection of any two of them.

Appendix C

Objectives: The main objective of this course is to equip students with the necessary techniques required to differentiate and integrate standard functions and to enable students to apply calculus techniques to solve applied problems.

Methodology: This is a one-semester course with five one-hour lectures and a two hour tutorial per week.

Course Outline

Limits: Definition, evaluation techniques (such as substitution, cancellation and by rationalizing of the numerator/denominator); indeterminate forms; rates of change; Average and instantaneous rates of change.

Differentiation: Concept and definition of a derivative. Differentiation of elementary functions; rules of differentiation (i.e. differentiation of sums, products, and quotients, the chain rule, generalized power rule, implicit differentiation and differentiation of parametric equations). Higher order derivatives. Application of derivatives: tangent line, relative maxima and minima, curve sketching, rates of change, L'Hopital's rule; differentials and approximations.

Integrations: Concept of integral as area under a curve and as anti-derivative, integration of elementary functions; integration techniques (i.e. by pattern recognition , substitution, integration by parts and by using partial fractions). Definite integrals.

Application: Area under and between curves, arc length of a curve, surface area and volume of solids of revolutions.

Appendix D

M222-6 ***Calculus II*** ***2001/2002***

Course Aims

This is a core course for students majoring in mathematics, computer science and/or physics. The aim of the course is to lay down the foundation of calculus to enable students to understand the dynamics of change in the study of any system, whether it is physical, biological, economic or social.

Course Content

Relations and functions. Limits of functions. Continuity and differentiability of functions. Sequences and series: Arithmetic and geometric sequences and series, convergence and divergence of series, Ratio and comparison tests, Absolute convergence. Improper integrals and indeterminate forms. L'Hopital's rule. The exponential function as a limit. Definition and calculus of hyperbolic functions. Successive differentiation. The power series. Maclaurian's series and Taylor's series. Expansion of functions in Maclaurian's and Taylor's series. Rolle's theorem. Lagrange's Mean Value Theorem. Funcitons of two variables and their partial derivatives. Euler's Theorem on homogeneous functions. Total differential coefficient. The Jacobian matrix. Relative maxima and minima. Change of variables in double and triple integrals. Ordinary differential equations of the first order.

Methodology

A full year course having six credit hours and with two one-hour lectures and a two-hour tutorial per week.

Assessment

The course work will be constituted by at least one assignment and two tests each semester in the ratio 3:7 respectively. At the end of the second semester a comprehensive three-hour examination will be taken and will constitute two thirds of the overall course grade.

Appendix E

Graduate Studies in Science Mathematics and Technology Education

Year 1 ***Questionnaire 1*** ***24th March 2004***

Student Name**……………………………………………………**
Student Number………………….

Group…………….. Tutorial day of the week……………………………….

Use the spaces provided to answer all questions.

Question 1

A car travels in a testing field. Its position is measured carefully.

The table shows how the displacement of the car changed with

time for a period of 4 seconds.

S (displacement in metres)	0	36.1	56.2	66.8	69.0	71.1	73.3	75.5	77.8
t (time in seconds)	0	2.0	3.0	3.5	3.6	3.7	3.8	3.9	4.0

In finding the average velocity of the car from say $t = 3.5$ to $t = 3.6$, we can use the formula

$$\text{Average velocity} = \frac{\text{displacement}}{\text{time elapsed}} = \frac{(69.0 - 66.8)}{(3.6 - 3.5)} \text{ m/s} = 22 \text{ m/s}$$

(i) What would be the average velocity between $t = 3.5$ and $t = 4.0$?

(ii) At an instant both the displacement and the time elapsed are zero. Does this mean that the car has stopped moving? Explain your answer.

(iii) The whole journey is made up of velocities that change from instant to instant. Does this mean that the car is not moving? Explain your answer.

(iv) Can we use the formula for the average velocity to obtain the answer for the velocity at an instant $t = 4$? Explain your answer.

(v) Is it possible from the average velocity values to obtain the instantaneous velocity? Explain your answer?

(vi) Which of the symbols in the box would represent:

i. Average velocity?

ii. Instantaneous velocity?

$\frac{\delta s}{\delta t}$, $\frac{\Delta s}{\Delta t}$, $\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$, $\frac{ds}{dt}$

(vii) What is the relationship between the given set of symbols (if any)?

(viii) What is the relationship between the average velocity and the instantaneous velocity (if any)?

(ix) What is your understanding of the expressions:

(a) "rate of change"

(b) "rate of change of *a* with respect to *b*"

(c) "Average rate of change"

(d) "Instantaneous rate of change"

Question 2

Two expressions concerning limits are given below.

Calculate (a) $\lim_{x \to 0} \frac{\sqrt{x^2+9}-3}{x^2}$ and (b) $\lim_{x \to \infty} \frac{\sqrt{x^2+9}-3}{x^2}$

Answer the questions that follow about expressions (a) and (b).

(i) Is it the same to find the limit of the given function as $x \to 0$ and as $x \to \infty$? Explain your answer.

(ii) In finding the limit in question (a) the number 0 is substituted for x in the functional part and the result obtained becomes 0/0. What conclusion can you draw from this result?

- The limit does not exist.

- The limit is 0.
- The limit is 1.
- It is an indeterminate state.
- The limit is ∞
- Any other, please specify.

Choose the option(s) that best describes your answer.

(iii) For question (b) write down any five numbers which you would substitute for x and explain why you think you have made an appropriate choice of numbers.

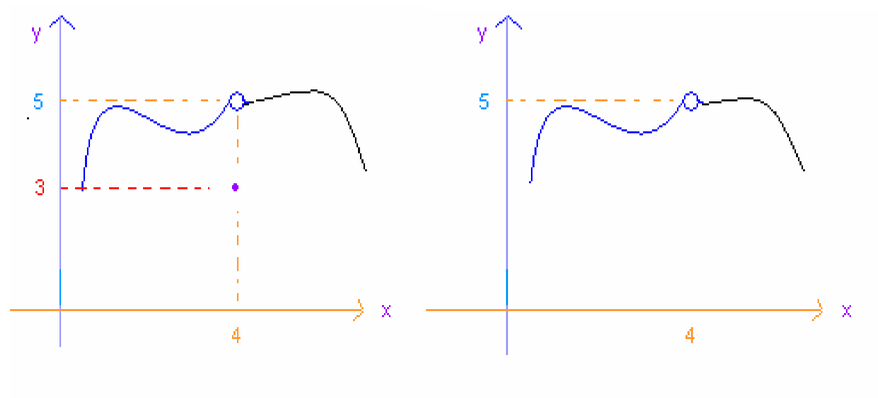
(iv) Calculate the limits of the function as given in (a) and (b).

(a)

(b)

Question 3

You have been asked to find the limits of functions below as x tends to 4 (if any). Which of the statements below would you agree with about the two functions?



(i) (ii)

(a) In diagram (ii) the limit does not exist since the function is not defined at $x = 4$.

(b) In diagram (i) the limit is 3 since it is the function value at $x = 4$.

(c) The two functions have the same limit since we are not concerned as to what happens at $x = 4$ but to values in its neighbourhood and their function values.

(d) The limits for the two functions cannot be obtained since the two functions are not defined at $x = 4$.

Why do you agree with the statement(s) you have chosen?

Question 4

How can we see if a function $y = f(x)$ has a limit L as x is approaching 0 ?

It is by:

7. Calculating y for $x = 0$, i.e. calculate $f(0)$
8. Calculating $f(1)$, $f(2)$, $f(3)$ and so on and observe the results
9. Calculating $f(x)$ for $x = 1/2, 1/4, 1/8$ and so on
10. Substituting x by 0 in the function formula, and calculate the value.
11. Substituting numbers that are very close to 0 for x in the formula and look for the value of y.

12. Substituting numbers that are very close to *0* for x in the formula and look for the value of *y* that is being approached as *x* values approach 0.

(Choose the option(s) that best describes your answer).

Why will you do so?

Question 5

The function $y = f(x)$ is calculated for values of x, and here are some results

x	$y = f(x)$
0.7	1
0.74	1.8
0.749	1.89
0.7499	1.899
0.74999	1.8999

(a) If this pattern continues what can you say about:

(iii) The number $f(x)$ is approaching?

(iv) The number x is approaching?

b) Complete the expression below so that it is true about the function represented by the table of values above:

(i) $\lim_{x \to ....} f(x) = ...$

(ii) After completing the expression above, write in words the meaning of the expression.

Question 6

Study the given expression and answer the questions that follow:

$$\lim_{x \to \infty} \frac{\sqrt{x^2+9}-3}{x}$$

(i) Write five numbers that you would use if you were to find the limit of the function using tables.

(ii) Does ∞ here represent a specific number or a set of numbers? Explain.

(iii) Calculate the limit as shown by the expression.

Question 7

a) Given the expression $\lim_{x \to a} \frac{1}{x}$

For what value of "*a*" does this limit exist? It is for:

(i) 0

(ii) ∞

(iii) "all real numbers"

(iv) "all real numbers except ……." (COMPLETE THE STATEMENT)

(Choose the option(s) that best describe your answer)

Why do you think so?

b) Given the expression $\lim_{x \to a} \frac{\sin x}{x}$

(i) For what value of *a* does this limit exist? It is for:

- 0
- ∞
- "all real numbers"

- “all real numbers except ….” (COMPLETE THE STATEMENT)

(*Choose the option*(*s*) *that best describe your answer*)

Why do you think so?

(ii) For small values of x (in radians) what is the relationship between the values of $\sin x$ and the values of x?

(iii) What does this relationship (if any) tell you about the limit of the given function as x tends to zero? Does the limit of the function exist as x tends to zero? If so, what is it?

(iv) For bigger values of x (in radians) what happens to the functional values of the given function?

(v) What does this tell you about the limit of the function as x values grow bigger and bigger? Does the limit exist? Explain your answer.

Appendix F

Graduate Studies in Science Mathematics and Technology Education

Year 2 ***Questionnaire 2*** ***23rd October 2004***

Student Name……………………………………………………

Student Number………………….

Use the spaces provided to answer all questions.

Question 1

Evaluate the following limits:

(a) $\lim_{n\to\infty}(-1)^n$

(b) $\lim_{n\to\infty}\frac{(-1)^n}{n}$

Question 2

Tick the option(s) that best describe your answer.

A sequence converges when:

- $a_n \to L$ *as* $n \to \infty$;
- $a_n \to L$ *as* $n \to -\infty$;
- *No specific value that* a_n *tends to as n tends to* ∞;
- *No specific value that* a_n *tends to as n tends to 0*;
- $a_n \to \infty$ *as* $n \to \infty$
- $|a_n - L| \to 0$ *as* $n \to \infty$;.

- $|a_{n+1} - a_n| \rightarrow 0$ *as* $n \rightarrow \infty$.

Why do you think so?

Question 3

Tick the option(s) that best describe your answer.

A sequence diverges when:

- $a_n \rightarrow L$ *as* $n \rightarrow \infty$;
- $a_n \rightarrow L$ *as* $n \rightarrow -\infty$;
- *No specific value that* a_n *tends to as n tends to* ∞;
- *No specific value that* a_n *tends to as n tends to 0*;
- $a_n \rightarrow \infty$ *as* $n \rightarrow \infty$
- $|a_n - L| \rightarrow 0$ *as* $n \rightarrow \infty$;.
- $|a_{n+1} - a_n| \rightarrow 0$ *as* $n \rightarrow \infty$.

.

Why do you think so?

Question 4

Find the limit of each sequence

(a) {½, ⅓, ¼, …}

(b) {-1, 1, -1, 1, -1, …}

(c) {1, 4, 9, 16, 25 ,…}

(d) {½, 0, ⅓, 0, ¼, 0, …}

Question 5

Tick the option(s) that best describe your answer.

The sequence $\{nr^n\}$ is convergent for:

- $-1 < r < 1$
- $-1 \le r \le 1$
- All real numbers
- All real numbers except 0

Why do you think so?

Question 6

Tick the option(s) that best describe your answer.

What does it mean to say that:

(a) $\lim_{n\to\infty} a_n = 8$

- $a_n \to 8$ *as* $n \to \infty$;
- $a_n \to \infty$ *as* $n \to 8$;

- *No specific value that a_n tends to as n tends to* ∞;
- *No specific value that a_n tends to as n tends to 0*;
- $a_n \to \infty$ *as* $n \to \infty$.

Why do you think so?

(b) $\lim_{n \to \infty} a_n = \infty$

- $a_n \to \infty$ *as* $n \to \infty$;
- $a_n \to \infty$ *as* $n \to -\infty$;
- *No specific value that a_n tends to as n tends to* ∞;
- *No specific value that a_n tends to as n tends to 0.*

Why do you think so?

Question 7

(a) What is the limit of the given sequence {3.1, 3.14, 3.141, 3.1415, 3.14159, …}?

Why do you think so?

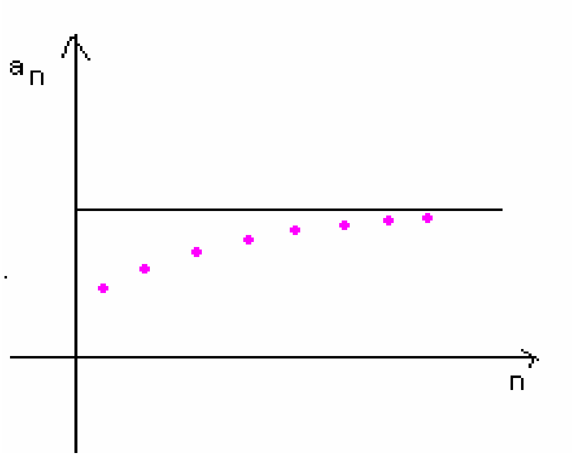
(b) Complete the equation below so that it is true about the sequence above.

$\lim_{n \to \ldots} a_n = \ldots\ldots\ldots$

Question 8

Which of the following graphs of sequences have limits? In each case explain how you obtained your answer.

(a)

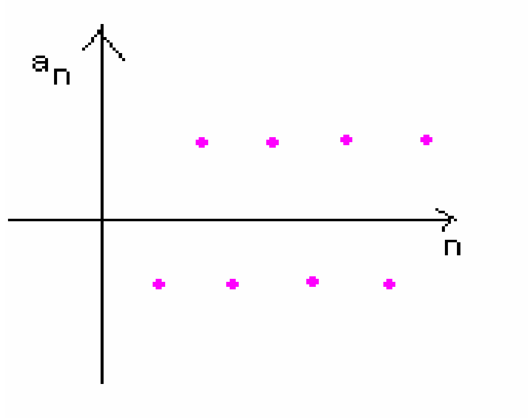


$$a_n = \frac{n}{n+1} \qquad , n = 1,2,3, \ldots$$

(a) The limit is:

I obtained this limit by

(b)



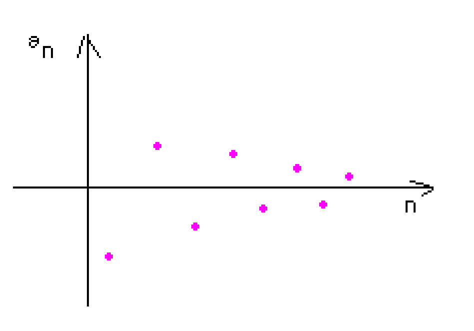
$a_n = (-1)^n, \, n = 1,2,3,$

...

The limit is:

I obtained this limit by

(c)



$$a_n = \frac{(-1)^n}{n}, n = 1,2,3, \ldots$$

The limit is:

I obtained this limit by