Aşağıdaki genelleştirilmiş integrallerin yakınsaklık durumunu inceleyiniz.

(a) $\displaystyle\int_{1}^{\infty} \frac{x-\sin x}{x^3}\,\mathrm{d}x$

(b) $\displaystyle\int_{0}^{1} \frac{x+2}{(x+1)(x-1)^2}\,\mathrm{d}x$

ÇÖZÜM:

(a) $\displaystyle\int_{1}^{\infty} \frac{x-\sin x}{x^3}\,\mathrm{d}x$

$$\begin{aligned}
-1 \leqslant &\quad \sin x \quad \leqslant 1\\
-1 \leqslant &\ -\sin x \ \leqslant 1\\
x-1 \leqslant &\ x-\sin x \leqslant x+1\\
\frac{x-1}{x^3} \leqslant &\ \frac{x-\sin x}{x^3} \leqslant \frac{x+1}{x^3}
\end{aligned}$$

$(1,\infty)$ aralığında, her üç fonksiyonda pozitiftir.

$$0 \leqslant \frac{x-1}{x^3} \leqslant \frac{x-\sin x}{x^3} \leqslant \frac{x+1}{x^3}$$

Eğer, $\displaystyle\int_{1}^{\infty} \frac{x+1}{x^3}\,\mathrm{d}x$ yakınsak ise, $\displaystyle\int_{1}^{\infty} \frac{x-\sin x}{x^3}\,\mathrm{d}x$ de yakınsaktır.

$$\int_{1}^{\infty} \frac{x+1}{x^3}\,\mathrm{d}x = \int_{1}^{\infty} \left(x^{-2}+x^{-3}\right)\,\mathrm{d}x = -\frac{1}{x} - \frac{1}{2x^2}\bigg|_{1}^{\infty} = 0 - \left(-\frac{3}{2}\right) = \frac{3}{2}$$

$\displaystyle\int_{1}^{\infty} \frac{x+1}{x^3}\,\mathrm{d}x = \frac{3}{2}$ yakınsak olduğundan, $\displaystyle\int_{1}^{\infty} \frac{x-\sin x}{x^3}\,\mathrm{d}x$ de yakınsaktır.

(b) $\displaystyle\int_{0}^{1} \frac{x+2}{(x+1)(x-1)^2}\,\mathrm{d}x$

$(0,1)$ aralığında, $\displaystyle\frac{x+2}{(x+1)(x-1)^2} > 0$ dır.

$\displaystyle\frac{x+2}{(x+1)(x-1)^2}$ fonksiyonunun payının 1 eksiği olan $\displaystyle\frac{x+1}{(x+1)(x-1)^2}$ fonksiyonunu alalım.

$$0 < \frac{x+1}{(x+1)(x-1)^2} < \frac{x+2}{(x+1)(x-1)^2}$$

Eğer, $\int\limits_0^1 \frac{x+1}{(x+1)(x-1)^2}\,\mathrm{d}x$ ıraksak ise, $\int\limits_0^1 \frac{x+2}{(x+1)(x-1)^2}\,\mathrm{d}x$ de ıraksaktır.

$$\int\limits_0^1 \frac{\cancel{x+1}}{(\cancel{x+1})(x-1)^2}\,\mathrm{d}x = \int\limits_0^1 \frac{1}{(x-1)^2}\,\mathrm{d}x = -\frac{1}{x-1}\Bigg|_0^1 = \underbrace{\left(-\frac{1}{(1^-)-1}\right)}_{+\infty} - \underbrace{\left(-\frac{1}{0-1}\right)}_{1} = +\infty$$

$\int\limits_0^1 \frac{x+1}{(x+1)(x-1)^2}\,\mathrm{d}x = \infty$ ıraksak olduğundan, $\int\limits_0^1 \frac{x+2}{(x+1)(x-1)^2}\,\mathrm{d}x$ de ıraksaktır.