In the early development of calculus almost all functions that were dealt with were continuous and there was no real need at that time for a penetrating look into the exact meaning of continuity. It was not until late in the 18th Century that discontinuous functions began appearing in connection with various kinds of physical problems. In particular, the work of J. B. J. Fourier (1758-1830) on the theory of heat forced mathematicians of the early 19th Century to examine more carefully the exact meaning of such concepts as ***function*** and ***continuity.*** Although the meaning of the word "continuous" seems intuitively clear to most people, it is not obvious how a good definition of this idea should be formulated. One popular dictionary explains continuity as follows :

Continuity: Quality or state of being continuous.

Continuous: Having continuity of parts.

Trying to learn the meaning of continuity from these two statements alone is like trying to learn Chinese with only a Chinese dictionary. A satisfactory mathematical definition of continuity, expressed entirely in terms of properties of the real-number system, was first formulated in 1821 by the French mathematician, Augustin-Louis Cauchy (1789-1857). His definition, which is still used today, is most easily explained in terms of the limit concept to which we turn now.

3.2 The defmition of the limit of a function

Let f be a function defined in some open interval containing a point p, although we do not insist that f be defined at the point p itself. Let A be a real number. The equation

$$\lim_{x \to p} f(x) = A$$

is read: "The limit off(x), as x approaches p, is equal to A," or "$f(x)$ approaches A as x approaches p." It is also written without the limit symbol, as follows:

$$f(x) \to A \qquad \text{as} \quad x \to p\,.$$

This symbolism is intended to convey the idea that we can make $f(x)$ as close to A as we please, provided we choose x sufficiently close to p.

Our first task is to explain the meaning of these symbols entirely in terms of real numbers. We shall do this in two stages. First we introduce the concept of a ***neighborhood*** of a point, then we define limits in terms of neighborhoods.

DEFINITION OF NEIGHBORHOOD OF A POINT. *Any open interval containing a point p as its midpoint is called a neighborhood of p.*

Notation. We denote neighborhoods by $N(p)$, $N_1(p)$, $N_2(p)$, etc. Since a neighborhood $N(p)$ is an open interval symmetric about p, it consists of all real x satisfying $p - r < x < p + r$ for some $r > 0$. The positive number r is called the ***radius*** of the neighborhood. We designate $N(p)$ by $N(p; r)$ if we wish to specify its radius. The inequalities $p - r < x < p + r$ are equivalent to $-r < x - p < r$, and to $|x - p| < r$. Thus, $N(p; r)$ consists of all points x whose distance from p is less than r.

In the next definition, we assume that A is a real number and thatfis a function defined on some neighborhood of a point p (except possibly at p). The function f may also be defined at p but this is irrelevant in the definition.

DEFINITION OF LIMIT OF A FUNCTION. *The symbolism*

$$\lim_{x \to p} f(x) = A \qquad [or\ f(x) \to A \qquad as \quad x - p]$$

means that for every neighborhood $N_1(A)$ there is some neighborhood N,(p) such that

$$(3.1) \qquad f(x) \in N_1(A) \qquad whenever\ x \in N_2(p) \qquad and\ \ x \neq p\,.$$

The first thing to note about this definition is that it involves *two* neighborhoods, *N,(A)* and *N,(p).* The neighborhood *N,(A)* is specified *first;* it tells us how close we wish $f(x)$ to

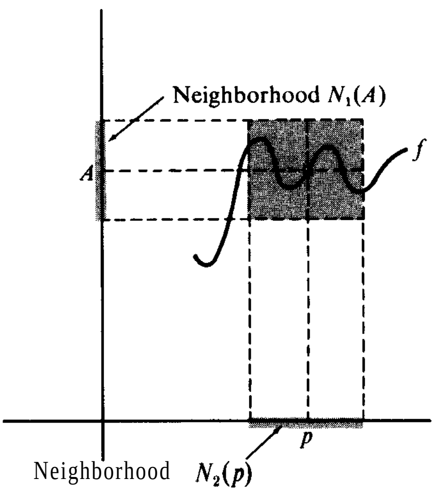

FIGURE 3.2 Here $\lim_{x\to p} f(x) = A$, but there is no assertion about f at p.

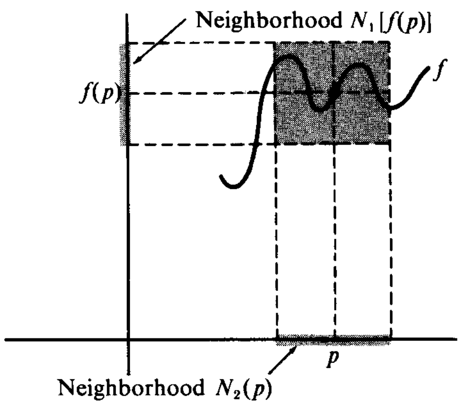

FIGURE 3.3 Here f is defined at p and $\lim_{x\to p} f(x) = f(p)$, hence f is continuous at p.

be to the limit A. The second neighborhood, *N,(p),* tells us how close x should be to p so that $f(x)$ will be within the first neighborhood *N,(A).* The essential part of the definition is that, for *every $N_1(A)$, no matter how small,* there is *some* neighborhood *N,(p)* to satisfy (3.1). In general, the neighborhood *N,(p)* will depend on the choice of *N,(A).* A neighborhood *N,(p)* that works for one particular $N_1(A)$ will also work, of course, for every larger *N,(A),* but it may not be suitable for any smaller *N,(A).*

The definition of limit may be illustrated geometrically as in Figure 3.2. A neighborhood $N_1(A)$ is shown on the y-axis. A neighborhood *N,(p)* corresponding to $N_1(A)$ is shown on the x-axis. The shaded rectangle consists of all points (x, y) for which x $\in$ *N,(p)* and y $\in N_1(A)$. The definition of limit asserts that the entire graph offabove the interval *N,(p)* lies within this rectangle, except possibly for the point on the graph above p itself.

The definition of limit can also be formulated in terms of the *radii* of the neighborhoods $N_1(A)$ and $N_2(p)$. It is customary to denote the radius of $N_1(A)$ by ϵ (the Greek letter *epsilon)* and the radius of $N_2(p)$ by δ (the Greek letter *delta).* The statement $f(x) \in N_1(A)$ is equivalent to the inequality $|f(x) - A| < \epsilon$, and the statement $x \in N_2(p)$, $x \neq p$, is equivalent to the inequalities $0 < |x - p| < \delta$. Therefore, the definition of limit can also be expressed as follows :

The symbol $\lim_{x \to p} f(x) = A$ *means that for every* $\epsilon > 0$, *there is a* $\delta > 0$ *such that*

$$|f(x) - A| < \epsilon \quad \text{whenever} \quad 0 < |x - p| < \delta . \tag{3.2}$$

We note that the three statements,

$$\lim_{x \to p} f(x) = A , \qquad \lim_{x \to p} (f(x) - A) = 0 , \qquad \lim_{x \to p} |f(x) - A| = 0 ,$$

are all equivalent. This equivalence becomes apparent as soon as we write each of these statements in the ϵ, δ-terminology (3.2).

In dealing with limits as $x \to p$, we sometimes find it convenient to denote the difference $x - p$ by a new symbol, say h, and to let $h \to 0$. This simply amounts to a change in notation, because, as can be easily verified, the following two statements are equivalent:

$$\lim_{x \to p} f(x) = A , \qquad \lim_{h \to 0} f(p + h) = A .$$

EXAMPLE 1. *Limit of a constant function.* Let $f(x) = c$ for all x. It is easy to prove that for every p, we have $\lim_{x \to p} f(x) = c$. In fact, given any neighborhood $N_1(c)$, relation (3.1) is trivially satisfied for any choice of $N_2(p)$ because $f(x) = c$ for all x and $c \in N_1(c)$ for all neighborhoods $N_1(c)$. In limit notation, we write

$$\lim_{x \to p} c = c .$$

EXAMPLE 2. *Limit of the identity function.* Here $f(x) = x$ for all x. We can easily prove that $\lim_{x \to p} f(x) = p$. Choose any neighborhood $N_1(p)$ and take $N_2(p) = N_1(p)$. Then relation (3.1) is trivially satisfied. In limit notation, we write

$$\lim_{x \to p} x = p .$$

"One-sided" limits may be defined in a similar way. For example, if $f(x) \to A$ as $x \to p$ through values greater than p, we say that A is the *right-hand limit of f* at p, and we indicate this by writing

$$\lim_{x \to p+} f(x) = A .$$

In neighborhood terminology this means that for every neighborhood $N_1(A)$, there is some neighborhood $N_2(p)$ such that

$$f(x) \in N_1(A) \quad \text{whenever} \quad x \in N_2(p) \quad \text{and} \quad x > p. \tag{3.3}$$

Left-hand limits, denoted by writing $x \to p-$, are similarly defined by restricting x to values less than *p.*

Iffhas a limit A at *p,* then it also has a right-hand limit and a left-hand limit at *p,* both of these being equal to A. But a function can have a right-hand limit at *p* different from the left-hand limit, as indicated in the next example.

EXAMPLE 3. Letf(x) = [x] for all x, and let *p* be any integer. For x near *p,* $x < p$, *we* have f(x) = $p - 1$, and for x near *p,* $x > p$, we have f(x) = *p.* Therefore we see that

$$\lim_{x\to p-} f(x) = p - 1 \qquad \text{and} \qquad \lim_{x\to p+} f(x) = p\,.$$

In an example like this one, where the right- and left-hand limits are unequal, the limit of fat *p does not exist.*

EXAMPLE 4. Let f(x) = $1/x^2$ if $x \neq 0$, and let $f(0) = 0$. The graph off near zero is shown in Figure 3.1(b). In this example,ftakes arbitrarily large values near 0 so it has no right-hand limit and no left-hand limit at 0. To prove rigorously that there is no real number A such that $\lim_{x\to 0+} f(x) = A$, we may argue as follows: Suppose there were such an A, say $A \geq 0$. Choose a neighborhood N,(A) of length 1. In the interval $0 < x < 1/(A + 2)$, we have $f(x) = 1/x^2 > (A + 2)^2 > A + 2$, so $f(x)$ cannot lie in the neighborhood N,(A). Thus, every neighborhood N(0) contains points $x > 0$ for which $f(x)$ is outside N,(A), so (3.3) is violated for this choice of N,(A). Hencefhas no right-hand limit at 0.

EXAMPLE 5. Let f(x) = 1 if $x \neq 0$, and let f(0) = 0. This function takes the constant value 1 everywhere except at 0, where it has the value 0. Both the right- and left-hand limits are 1 at every point *p,* so the limit off(x), as x approaches *p,* exists and equals 1. Note that the limit of f is 1 at the point 0, even though $f(0) = 0$.

3.3 The definition of continuity of a function

In the definition of limit we made no assertion about the behavior off at the point *p* itself. Statement (3.1) refers to those $x \neq p$ which lie in $N_2(p)$, so it is not necessary that f be defined at *p.* Moreover, even if f is defined at *p,* its value there need not be equal to the limit A. However, if it happens thatf is defined atp and if it also happens thatf(p) = A, then we say the function f is continuous at *p.* In other words, we have the following definition.

DEFINITION OF CONTINUITY OF A FUNCTION AT A POINT. *A function f is said to be continuous at a point p if*

(a) *fis defined at p, and*

(b) $\lim_{x\to p} f(x) = f(p)$.

This definition can also be formulated in terms of neighborhoods. A function f is continuous at *p* if for every neighborhood $N_1[f(p)]$ there is a neighborhood $N_2(p)$ such that

$$f(x) \in N_1[f(p)] \qquad \text{whenever } x \in N_2(p)\,. \tag{3.4}$$