If ϵ is any positive number, *no matter how small*, we must be able to ensure that $|f(x) - L| < \epsilon$ by restricting x to be *close enough to* (but not equal to) a. How close is close enough? It is sufficient that the distance $|x - a|$ from x to a be less than a positive number δ that depends on ϵ. (See Figure 1.36.) If we can find such a δ for any positive ϵ, we are entitled to conclude that $\lim_{x\to a} f(x) = L$.

DEFINITION 8

A formal definition of limit

We say that $f(x)$ **approaches the limit** L as x **approaches** a, and we write

$$\lim_{x\to a} f(x) = L,$$

if the following condition is satisfied:
for every number $\epsilon > 0$ there exists a number $\delta > 0$, possibly depending on ϵ, such that if $0 < |x - a| < \delta$, then x belongs to the domain of f and

$$|f(x) - L| < \epsilon.$$

The formal definition of limit does not tell you how to find the limit of a function, but it does enable you to verify that a suspected limit is correct. The following examples show how it can be used to verify limit statements for specific functions. The first of these gives a formal verification of the two limits found in Example 3 of Section 1.2.

EXAMPLE 2 **(Two important limits)** Verify: (a) $\lim_{x\to a} x = a$ and (b) $\lim_{x\to a} k = k$ (k = constant).

Solution

(a) Let $\epsilon > 0$ be given. We must find $\delta > 0$ so that

$$0 < |x - a| < \delta \qquad \text{implies} \qquad |x - a| < \epsilon.$$

Clearly, we can take $\delta = \epsilon$ and the implication above will be true. This proves that $\lim_{x\to a} x = a$.

(b) Let $\epsilon > 0$ be given. We must find $\delta > 0$ so that

$$0 < |x - a| < \delta \qquad \text{implies} \qquad |k - k| < \epsilon.$$

Since $k - k = 0$, we can use any positive number for δ and the implication above will be true. This proves that $\lim_{x\to a} k = k$.

EXAMPLE 3 Verify that $\lim_{x\to 2} x^2 = 4$.

Solution Here $a = 2$ and $L = 4$. Let ϵ be a given positive number. We want to find $\delta > 0$ so that if $0 < |x - 2| < \delta$, then $|f(x) - 4| < \epsilon$. Now

$$|f(x) - 4| = |x^2 - 4| = |(x + 2)(x - 2)| = |x + 2||x - 2|.$$

We want the expression above to be less than ϵ. We can make the factor $|x - 2|$ as small as we wish by choosing δ properly, but we need to control the factor $|x + 2|$ so that it does not become too large. If we first assume $\delta \le 1$ and require that $|x - 2| < \delta$, then we have

$$|x - 2| < 1 \quad \Rightarrow \quad 1 < x < 3 \quad \Rightarrow \quad 3 < x + 2 < 5$$
$$\Rightarrow \quad |x + 2| < 5.$$

Hence,

$$|f(x) - 4| < 5|x - 2| \quad \text{if} \quad |x - 2| < \delta \leq 1.$$

But $5|x - 2| < \epsilon$ if $|x - 2| < \epsilon/5$. Therefore, if we take $\delta = \min\{1, \epsilon/5\}$, the *minimum* (the smaller) of the two numbers 1 and $\epsilon/5$, then

$$|f(x) - 4| < 5|x - 2| < 5 \times \frac{\epsilon}{5} = \epsilon \quad \text{if} \quad |x - 2| < \delta.$$

This proves that $\lim_{x\to 2} f(x) = 4$.

Using the Definition of Limit to Prove Theorems

We do not usually rely on the formal definition of limit to verify specific limits such as those in the two examples above. Rather, we appeal to general theorems about limits, in particular Theorems 2–4 of Section 1.2. The definition is used to prove these theorems. As an example, we prove part 1 of Theorem 2, the *Sum Rule*.

EXAMPLE 4 **(Proving the rule for the limit of a sum)** If $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = M$, prove that $\lim\limits_{x\to a}\big(f(x) + g(x)\big) = L + M$.

Solution Let $\epsilon > 0$ be given. We want to find a positive number δ such that

$$0 < |x - a| < \delta \quad \Rightarrow \quad \big|\big(f(x) + g(x)\big) - (L + M)\big| < \epsilon.$$

Observe that

$$\begin{aligned} &\big|\big(f(x)+g(x)\big) - (L + M)\big| && \text{Regroup terms.} \\ &\qquad = \big|\big(f(x) - L\big) + \big(g(x) - M\big)\big| && \text{(Use the triangle inequality:} \\ & && |a + b| \leq |a| + |b|\text{).} \\ &\qquad \leq |f(x) - L| + |g(x) - M|. \end{aligned}$$

Since $\lim_{x\to a} f(x) = L$ and $\epsilon/2$ is a positive number, there exists a number $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \quad \Rightarrow \quad |f(x) - L| < \epsilon/2.$$

Similarly, since $\lim_{x\to a} g(x) = M$, there exists a number $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \quad \Rightarrow \quad |g(x) - M| < \epsilon/2.$$

Let $\delta = \min\{\delta_1, \delta_2\}$, the smaller of δ_1 and δ_2. If $0 < |x - a| < \delta$, then $|x - a| < \delta_1$, so $|f(x) - L| < \epsilon/2$, and $|x - a| < \delta_2$, so $|g(x) - M| < \epsilon/2$. Therefore,

$$\big|\big(f(x) + g(x)\big) - (L + M)\big| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This shows that $\lim_{x\to a}\big(f(x) + g(x)\big) = L + M$.

Other Kinds of Limits

The formal definition of limit can be modified to give precise definitions of one-sided limits, limits at infinity, and infinite limits. We give some of the definitions here and leave you to supply the others.