Soru

$\lim_{x\to 0}\left(\cot^2 x-\frac{1}{x^2}\right)$ değerini bulunuz.

Çözüm

I. yol (Hospital Kuralı ile)

$$\ell=\lim_{x\to 0}\left(\cot^2 x-\frac{1}{x^2}\right)=\lim_{x\to 0}\left(\frac{\cos^2 x}{\sin^2 x}-\frac{1}{x^2}\right)$$
$$=\lim_{x\to 0}\left(\frac{1-\sin^2 x}{\sin^2 x}-\frac{1}{x^2}\right)$$
$$=\lim_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}-1\right)$$
$$=-1+\lim_{x\to 0}\left(\frac{x^2-\sin^2 x}{x^2\cdot\sin^2 x}\right)$$

Burada Hospital Kuralını hemen uygulamak işlemleri uzatır.
Parantez içini parçalayalım:

$$\ell=-1+\lim_{x\to 0}\left(\frac{x-\sin x}{x^3}\cdot\frac{x}{\sin x}\cdot\frac{x+\sin x}{\sin x}\right)$$
$$=-1+\lim_{x\to 0}\left(\frac{x-\sin x}{x^3}\right)\cdot\lim_{x\to 0}\left(\frac{x}{\sin x}\right)\cdot\lim_{x\to 0}\left(\frac{x+\sin x}{\sin x}\right)$$
$$=-1+\lim_{x\to 0}\left(\frac{1-\cos x}{3x^2}\right)\cdot 1\cdot 2$$
$$=-1+2\cdot\lim_{x\to 0}\left(\frac{\sin x}{6\cdot x}\right)$$
$$=-\frac{2}{3}$$

elde edilir.

II. yol

$$\ell=\lim_{x\to 0}\left(\cot^2 x-\frac{1}{x^2}\right)=\lim_{x\to 0}\left(\frac{\cos^2 x}{\sin^2 x}-\frac{1}{x^2}\right)$$
$$=\lim_{x\to 0}\left(\frac{1-\sin^2 x}{\sin^2 x}-\frac{1}{x^2}\right)$$
$$=\lim_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}-1\right)$$
$$=-1+\lim_{x\to 0}\left(\frac{x^2-\sin^2 x}{x^2\cdot\sin^2 x}\right)$$

Parantez içini I. yoldaki gibi parçalayalım:

$$\ell=-1+\lim_{x\to 0}\left(\frac{x-\sin x}{x^3}\cdot\frac{x}{\sin x}\cdot\frac{x+\sin x}{\sin x}\right);$$
$$\ell=-1+\underbrace{\lim_{x\to 0}\left(\frac{x-\sin x}{x^3}\right)}_{k}\cdot\underbrace{\lim_{x\to 0}\left(\frac{x}{\sin x}\right)}_{1}\cdot\underbrace{\lim_{x\to 0}\left(\frac{x+\sin x}{\sin x}\right)}_{2}$$

$$k=\lim_{x\to 0}\frac{x-\sin x}{x^3}=\lim_{x\to 0}\frac{x-3\cdot\sin\frac{x}{3}+4\cdot\sin^3\frac{x}{3}}{x^3}$$

$$\Rightarrow \lim_{x\to 0}\frac{x-\sin x}{x^3}=\lim_{x\to 0}\frac{x-3\cdot\sin\frac{x}{3}}{x^3}+\lim_{x\to 0}\frac{4\cdot\sin^3\frac{x}{3}}{x^3}\quad(1)$$

İlk terimde $x=3t$ dönüşümü yaparsak,

$$\lim_{x\to 0}\frac{x-\sin x}{x^3}=k;$$
$$\lim_{x\to 0}\frac{x-3\cdot\sin\frac{x}{3}}{x^3}=\lim_{t\to 0}\frac{3t-3\cdot\sin t}{27.t^3}$$
$$=\frac{3}{27}\cdot\lim_{t\to 0}\frac{t-\sin t}{t^3}=\frac{1}{9}\cdot k;$$
$$\lim_{x\to 0}\frac{4\cdot\sin^3\frac{x}{3}}{x^3}=\lim_{x\to 0}\frac{4\cdot\sin^3\frac{x}{3}}{27\cdot(\frac{x}{3})^3}=\frac{4}{27}$$

olur.
Bu değerler (1)'de yerlerine konulursa;

$$k=\frac{1}{9}\cdot k+\frac{4}{27}$$
$$\Rightarrow k=\frac{1}{6}\quad \text{bulunur.}$$

$$\ell=-1+\frac{1}{6}\cdot 1\cdot 2$$
$$\Rightarrow \ell=-\frac{2}{3}\ \text{olur.}$$