Örnek Problem

R'den R'ye f ve g fonksiyonları,

$$f(x)=\begin{cases}(x-2)^2+1, & x<2 \text{ ise}\\ (x-2)^2-1, & x\geq 2 \text{ ise}\end{cases}$$ ve

$$g(x)=\begin{cases}x+1, & x<1 \text{ ise}\\ 1-x, & x\geq 1 \text{ ise}\end{cases}$$

kuralları ile verilmiştir.

a. $(fog)(x)=?$

b. $(gof)(x)=?$

c. $(fog)(x-2)=?$

d. $(gof)(2x)=?$

e. $y=f(x)$, $y=g(x)$, $y=(fog)(x)$, $y=(gof)(x)$, $y=(fog)(x-2)$ ve $y=(gof)(2x)$ fonksiyonlarının grafiklerini çiziniz.

f. $\lim\limits_{x\to 2^+}(fog)(x)=?$

g. $\lim\limits_{x\to 2^-}(gof)(x)=?$

h. $\lim\limits_{x\to 1^+}(fog)(x-2)=?$

k. $\lim\limits_{x\to 0^-}(gof)(2x)=?$

Çözüm

a. $(fog)(x)=?$

$x<1$ ise $g(x)=x+1$ olur.

$$(fog)(x)=f(x+1)=\begin{cases}(x+1-2)^2+1, & x+1<2 \text{ ise}\\ (x+1-2)^2-1, & x+1\geq 2 \text{ ise}\end{cases}$$

$$\Rightarrow (fog)(x)=\begin{cases}(x-1)^2+1, & x<1 \text{ ise}\\ (x-1)^2-1, & x\geq 1 \text{ ise}\end{cases}$$

Demek ki; $x<1$ iken, $(fog)(x)=(x-1)^2+1 \quad (1)$

olacaktır.

$x\geq 1$ ise $g(x)=1-x$ olur.

$$(fog)(x)=f(1-x)=\begin{cases}(1-x-2)^2+1, & 1-x<2 \text{ ise}\\ (1-x-2)^2-1, & 1-x\geq 2 \text{ ise}\end{cases}$$

$$\Rightarrow (fog)(x)=\begin{cases}(x+1)^2+1, & x>-1 \text{ ise}\\ (x+1)^2-1, & x\leq -1 \text{ ise}\end{cases}$$

Demek ki; $x\geq 1$ iken, $(fog)(x)=(x+1)^2+1 \quad (2)$

olacaktır.

(1) ve (2) birleştirilirse,

$$fog(x)=\begin{cases}(x-1)^2+1, & x<1 \text{ ise}\\ (x+1)^2+1, & x\geq 1 \text{ ise}\end{cases}$$

elde edilir.

b. $(gof)(x)=?$

$x<2$ ise $f(x)=(x-2)^2+1$ olur.

$$(gof)(x)=g\left[(x-2)^2+1\right]$$

$$=\begin{cases}\left[(x-2)^2+1\right]+1, & (x-2)^2+1<1 \text{ ise}\\ 1-\left[(x-2)^2+1\right], & (x-2)^2+1\geq 1 \text{ ise}\end{cases}$$

Demek ki; $x<2$ iken, $(gof)(x)=-(x-2)^2 \quad (1)$

olacaktır.

$x\geq 2$ ise $f(x)=(x-2)^2-1$ olur.

$$(gof)(x)=g\left[(x-2)^2-1\right]$$

$$=\begin{cases}\left[(x-2)^2-1\right]+1, & (x-2)^2-1<1 \text{ ise}\\ -\left[(x-2)^2-1\right]+1, & (x-2)^2-1\geq 1 \text{ ise}\end{cases}$$

Demek ki; $x\geq 2$ iken,

$$(gof)(x)=\begin{cases}(x-2)^2, & 2\leq x<2+\sqrt{2} \text{ ise}\\ 2-(x-2)^2, & x\geq 2+\sqrt{2} \text{ ise}\end{cases} \quad (2)$$

olacaktır.

(1) ve (2) birleştirilirse,

$$(gof)(x)=\begin{cases}-(x-2)^2, & x<2 \text{ ise}\\ (x-2)^2, & 2\le x<2+\sqrt{2} \text{ ise}\\ 2-(x-2)^2, & x\ge 2+\sqrt{2} \text{ ise}\end{cases}$$

elde edilir.

c. $(fog)(x-2)=?$

$$fog(x)=\begin{cases}(x-1)^2+1, & x<1 \text{ ise}\\ (x+1)^2+1, & x\ge 1 \text{ ise}\end{cases}$$

$$\Rightarrow fog(x-2)=\begin{cases}(x-2-1)^2+1, & x-2<1 \text{ ise}\\ (x-2+1)^2+1, & x-2\ge 1 \text{ ise}\end{cases}$$

$$\Rightarrow fog(x-2)=\begin{cases}(x-3)^2+1, & x<3 \text{ ise}\\ (x-1)^2+1, & x\ge 3 \text{ ise}\end{cases}$$

olur.

d. $(gof)(2x)=?$

$$(gof)(x)=\begin{cases}-(x-2)^2, & x<2 \text{ ise}\\ (x-2)^2, & 2\le x<2+\sqrt{2} \text{ ise}\\ 2-(x-2)^2, & x\ge 2+\sqrt{2} \text{ ise}\end{cases}$$

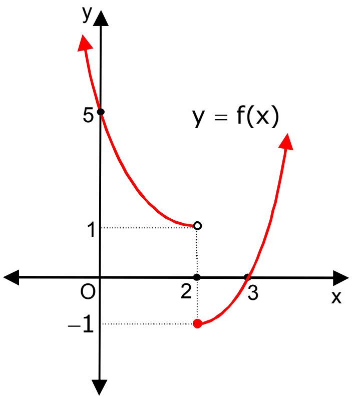
$$\Rightarrow (gof)(2x)=\begin{cases}-(2x-2)^2, & 2x<2 \text{ ise}\\ (2x-2)^2, & 2\le 2x<2+\sqrt{2} \text{ ise}\\ 2-(2x-2)^2, & 2x\ge 2+\sqrt{2} \text{ ise}\end{cases}$$

$$\Rightarrow (gof)(2x)=\begin{cases}-(2x-2)^2, & x<1 \text{ ise}\\ (2x-2)^2, & 1\le x<1+\dfrac{\sqrt{2}}{2} \text{ ise}\\ 2-(2x-2)^2, & x\ge 1+\dfrac{\sqrt{2}}{2} \text{ ise}\end{cases}$$

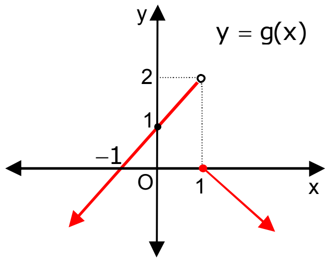
olur.

e.

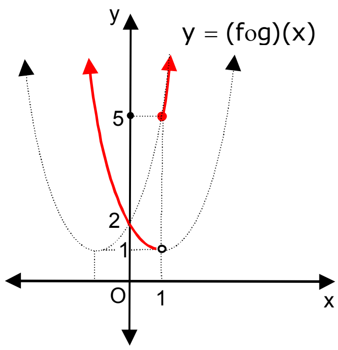
$$f(x)=\begin{cases}(x-2)^2+1, & x<2 \text{ ise}\\ (x-2)^2-1, & x\ge 2 \text{ ise}\end{cases}$$



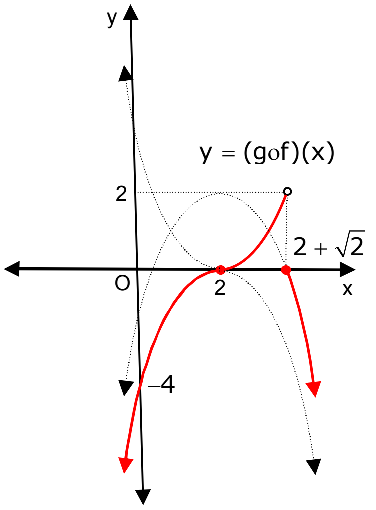
$$g(x)=\begin{cases}x+1, & x<1 \text{ ise}\\ 1-x, & x\ge 1 \text{ ise}\end{cases}$$



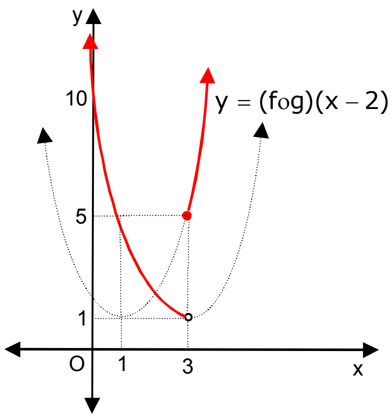
$$fog(x)=\begin{cases}(x-1)^2+1, & x<1 \text{ ise}\\ (x+1)^2+1, & x\ge 1 \text{ ise}\end{cases}$$



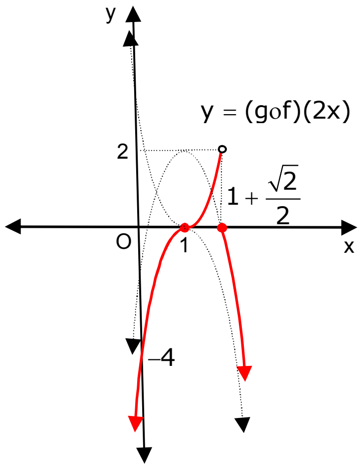
$$(gof)(x)=\begin{cases}-(x-2)^2, & x<2 \text{ ise}\\ (x-2)^2, & 2\le x<2+\sqrt{2} \text{ ise}\\ 2-(x-2)^2, & x\ge 2+\sqrt{2} \text{ ise}\end{cases}$$



$$fog(x-2)=\begin{cases}(x-3)^2+1, & x<3 \text{ ise}\\ (x-1)^2+1, & x\ge 3 \text{ ise}\end{cases}$$



$$(gof)(2x)=\begin{cases}-(2x-2)^2, & x<1 \text{ ise}\\ (2x-2)^2, & 1\le x<1+\frac{\sqrt{2}}{2} \text{ ise}\\ 2-(2x-2)^2, & x\ge 1+\frac{\sqrt{2}}{2} \text{ ise}\end{cases}$$



f. $\lim\limits_{x\to 2^+}(fog)(x)=?$

fog fonksiyonunun kuralı yazıldığında bu sorunun yanıtı apaçıktır. Bu tür sorular, genellikle, f ve g fonksiyonlarının ayrı ayrı kuralları ya da grafikleri üzerinden sorulur. Biz de, fog fonksiyonunun kuralını bilmiyormuş gibi yapalım:

$$f(x)=\begin{cases}(x-2)^2+1, & x<2 \text{ ise}\\ (x-2)^2-1, & x\ge 2 \text{ ise}\end{cases}$$

$$g(x)=\begin{cases}x+1, & x<1 \text{ ise}\\ 1-x, & x\ge 1 \text{ ise}\end{cases}$$

$$\lim_{x\to 2^+}(fog)(x)\to f\left(g\left(2^+\right)\right)\to f\left(-1^-\right)\to 10$$

$$\Rightarrow \lim_{x\to 2^+}(fog)(x)=10$$

g. $\lim_{x\to 2^-}(gof)(x)=?$

$$\lim_{x\to 2^-}(gof)(x)\to g\left(f\left(2^-\right)\right)\to f\left(1^+\right)\to 0$$
$$\Rightarrow \lim_{x\to 2^-}(gof)(x)=0$$

h. $\lim_{x\to 1^+}(fog)(x-2)=?$

$$\lim_{x\to 1^+}(fog)(x-2)\to f\left(g\left(-1^+\right)\right)\to f\left(0^-\right)\to 5$$
$$\Rightarrow \lim_{x\to 1^+}(fog)(x-2)=5$$

k. $\lim_{x\to 0^-}(gof)(2x)=?$

$$\lim_{x\to 0^-}(gof)(2x)\to g\left(f\left(0^-\right)\right)\to g\left(5^+\right)\to -4$$
$$\Rightarrow \lim_{x\to 0^-}(gof)(2x)=-4$$