INVERSE LAPLACE TRANSFORM OF e^{-2s}

The inverse Laplace transform of a function $F(s)$ is given by the integral

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}F(s)\,ds,$$

where c is a real number chosen greater than the real parts of all singularities of F . The argument t is assumed real. It is assumed that $F(s)$ is integrable on the vertical line. This is the case when it is the Laplace transform of reasonably smooth and bounded (or slowly growing) function $f(t)$.

If $F(s)$ is not integrable but does not grow too fast (technically, if it defines a *tempered distribution*), it is still possible to find its inverse transform as a tempered distribution. If the integral is divided into two parts

$$I_1(t) = \int_{c}^{c+i\infty} e^{st}F(s)\,ds,$$

and

$$I_2(t) = \int_{c}^{c-i\infty} e^{st}F(s)\,ds,$$

then the exponential factor e^{st} makes $I_1(t)$ and $I_2(t)$ convergent and analytic for t in the upper half and lower half plane, respectively. These functions have singularities on the real boundary, but they can be interpreted as distributions. Hence the difference of the integrals yields a well defined distribution which can be called the inverse Laplace transform of $F(s)$.

In the case $F(s) = e^{-2s}$ we may take $c = 0$, and the integrals are readily computed as

$$I_1(t) = \frac{-1}{t-2} \qquad (\mathrm{Im}(t) > 0),$$

$$I_2(t) = \frac{-1}{t-2} \qquad (\mathrm{Im}(t) < 0).$$

To find the boundary values one has to consider the integration of these functions along the lines $\mathbf{R} + i\varepsilon$ and $\mathbf{R} - i\varepsilon$. The upper line may be deformed to a contour that consists of the interval from $-\infty$ to $2-\varepsilon$, a half-circle in the upper half plane of center 2 and radius ε, and the interval from $2+\varepsilon$ to ∞.

In the limit $\varepsilon \to 0$ the distribution defined by the two intervals is the *principal part* of $-1/(t-2)$ while the half-circle in the negative direction gives $-\pi i$ times the residue. Hence the boundary value of $I_1(t)$ becomes

$$\mathrm{pp}\frac{-1}{t-2} + \pi i \delta(t-2)\,.$$

Similarly, the boundary value of $I_2(t)$ is

$$\mathrm{pp}\frac{-1}{t-2} - \pi i \delta(t-2)\,.$$

Taking the difference, the principal parts cancel and we are left with the inverse Laplace transform $f(t) = \delta(t-2)$.