## Short-cut Methods :

Consider the linear differential equation

$$f(D)\,y = R(x)$$

where (I) $R(x) = e^{ax}$

(II) $R(x) = \sin(ax+b)$ or $\cos(ax+b)$

(III) $R(x) = x^m$ where $m$ is a positive integer

(IV) $R(x) = e^{ax}\cdot V(x)$ where $V(x)$ is a function of $x$.

(V) $R(x) = x^m \cdot V(x)$ where $m$ is a positive integer & $V(x)$ is any function of $x$

☆ **Case I :** when $R(x) = e^{ax}$

$$P.I. = \frac{1}{f(D)} e^{ax} = \begin{cases} \dfrac{1}{f(a)} e^{ax} & ; \quad f(a) \neq 0 \\[2ex] x \dfrac{1}{f'(a)} e^{ax} & ; \quad f(a) = 0 \end{cases}$$

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**Note :** If $R(x) = \cosh x$ or $\sinh x$ or $k$ or $a^x$ then we can find P.I. by case (I) because

(i) $\cosh x = \dfrac{e^x + e^{-x}}{2}$ (ii) $\sinh x = \dfrac{e^x - e^{-x}}{2}$

(iii) $k = k \cdot e^{0x}$ (iv) $a^x = e^{x \log a}$

* **Example 1 :** Find the general solution of

$y''' - 3y'' + 3y' - y = 4e^t$ (G.T.U. Dec.2011) [04]

→ **Solution :** Given equation is (G.T.U. Jan. 2015) [04]

$$(D^3 - 3D^2 + 3D - 1)\,y = 4e^t$$

A.E. is $D^3 - 3D^2 + 3D - 1 = 0$

$D = 1$ satisfies it.

$\therefore$ $(D-1)$ is one of the factor of A.E

| | | | | |
|---|---|---|---|---|
| | 1 | −3 | 3 | −1 |
| 1 | 0 | 1 | −2 | 1 |
| | 1 | −2 | 1 | 0 |
| | ↑ $D^2$ | ↑ $D$ | ↑ constant | |

$\therefore$ $(D-1)(D^2 - 2D + 1) = 0$

$\therefore$ $(D-1)(D-1)^2 = 0$

$\therefore$ $(D-1)^3 = 0$ basis = {1, 1, 1}

$\therefore$ $D = 1, 1, 1$ for P I

C.F. $= (c_1 + c_2 t + c_3 t^2)\,e^t$ family = { 1 $e^t$ } → appears 3 times

Now $P.I. = \frac{1}{f(D)} R(x)$

$$= \frac{1}{D^3 - 3D^2 + 3D - 1} 4e^t$$

$$= 4 \frac{1}{D^3 - 3D^2 + 3D - 1} e^t$$

$$= 4 \cdot t \frac{1}{3D^2 - 6D + 3} e^t$$

$$= 4 \cdot t^2 \frac{1}{6D - 6} e^t$$

$$= 4 \cdot t^3 \frac{1}{6} e^t$$

$$= \frac{2}{3} t^3 e^t$$

Hence the general solution is

$$y = C.F. + P.I.$$

$$= (c_1 + c_2 t + c_3 t^2) e^t + \frac{2}{3} t^3 e^t$$

* Example 2 : Find the particular solution of $y = \frac{1}{(D+1)^2} \cosh x$ where $D = \frac{d}{dx}$. (G.T.U. May-12) [01]

~> Solution : $P.I. = \frac{1}{(D+1)^2} \boxed{\cosh x}$ ~~A A A~~

$$= \frac{1}{(D+1)^2} \boxed{\left[\frac{e^x + e^{-x}}{2}\right]}$$

$$= \frac{1}{2}\left[\frac{1}{(D+1)^2} e^x + \frac{1}{(D+1)^2} e^{-x}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(1+1)^2} e^x + x \frac{1}{2(D+1)} e^{-x}\right]$$

$$= \frac{1}{2}\left[\frac{1}{4} e^x + x^2 \frac{1}{2} e^{-x}\right]$$

$$= \frac{1}{8} e^x + \frac{1}{4} x^2 e^{-x}$$

 Prepared by : Prof. K. I. Patel (M.Sc., M. Phil) (M) 9427335039

**Case II** : when $R(x) = \sin(ax+b)$ or $\cos(ax+b)$

$$P.I. = \frac{1}{\phi(D^2)} \sin(ax+b) = \begin{cases} \frac{1}{\phi(-a^2)} \sin(ax+b) & ; \phi(-a^2) \neq 0 \\ x \frac{1}{\phi'(-a^2)} \sin(ax+b) & ; \phi(-a^2) = 0 \end{cases}$$

**Remark** :

(1) Replace $D^2$ by $-a^2$

(2) Replace $D^3$ by $D^2 \cdot D$

(3) Keep $D$ as it is. To get $D^2$ in the denominator rationalise the denominator and then replace $D^2$ by $-a^2$.

**Example 1 :** Solve $y'' + 4y = \sin 3x$

**Solution :** Given equation is

$$(D^2 + 4)y = \sin 3x$$

A.E. is $f(D) = 0$

$\therefore \quad D^2 + 4 = 0$

$\therefore \quad D^2 = -4 = 4i^2$

$\therefore \quad D = \pm 2i$

$$C.F. = e^{0x}(c_1 \cos 2x + c_2 \sin 2x)$$

$$= c_1 \cos 2x + c_2 \sin 2x$$

Now

$$P.I. = \frac{1}{f(D)} R(x)$$

$$= \frac{1}{D^2 + 4} \sin 3x \qquad (f(-3^2) \neq 0)$$

$$= \frac{1}{-(3)^2 + 4} \sin 3x \qquad (\because D^2 = -a^2 = -(3)^2$$

$$= -\frac{1}{5} \sin 3x$$

Hence the general solution is

$$y = C.F. + P.I.$$

$$= c_1 \cos 2x + c_2 \sin 2x - \frac{1}{5} \sin 3x$$

**Case III** : when $R(x) = x^m$ where $m$ is a positive integer.

$$P.I. = \frac{1}{f(D)} R(x)$$

$$= \frac{1}{f(D)} x^m$$

$$= \frac{1}{[1+\phi(D)]} x^m$$

$$= [1+\phi(D)]^{-1} x^m$$

Expand $[1+\phi(D)]^{-1}$ in ascending powers of $D$ up to the term containing $D^m$ by binomial Theorem.

Remark :

(1) we always take constant term common from the denominator and use the formulae

(i) $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \cdots\cdots$

(ii) $(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \cdots\cdots$

(iii) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots\cdots$

(iv) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots\cdots$

(v) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \cdots\cdots$

(2) If constant term is absent in the denominator, then minimum power of $D$ is taken common from the denominator

* **Example 3 :** Solve $y'' + 2y' + 3y = 2x^2$

(G.T.U. Jan. 2015) [07]

→ **Solution :** Given equation is

$$(D^2 + 2D + 3)y = 2x^2$$

A.E. is $f(D) = 0$

$$\therefore \quad D^2 + 2D + 3 = 0$$

$$a = 1, \quad b = 2, \quad c = 3$$

$$\Delta = b^2 - 4ac$$

$$= 4 - 4(1)(3)$$

$$= -8$$

$$= 8i^2$$

$$\sqrt{\Delta} = 2\sqrt{2}i$$

$$\therefore \quad D = \frac{-b \pm \sqrt{\Delta}}{2a}$$

$$= \frac{-2 \pm 2\sqrt{2}i}{2}$$

$$= -1 \pm \sqrt{2}i$$

$$\text{C.F.} = e^{-x}\left(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x\right)$$

$$\text{P.I.} = \frac{1}{f(D)} R(x)$$

$$= \frac{1}{D^2 + 2D + 3} 2x^2$$

$$= \frac{1}{3\left(\frac{D^2 + 2D}{3} + 1\right)} 2x^2$$

$$= \frac{1}{3}\left[\left(1 + \frac{2D + D^2}{3}\right)^{-1}\right] 2x^2$$

$$= \frac{1}{3}\left[1 - \left(\frac{2D + D^2}{3}\right) + \left(\frac{2D + D^2}{3}\right)^2 - \cdots\cdots\right] 2x^2$$

$$= \frac{1}{3}\left[1 - \frac{2}{3}D - \frac{1}{3}D^2 + \frac{4}{9}D^2 \cdots\cdots\right] 2x^2$$

$$= \frac{2}{3}\left[1 - \frac{2}{3}D + \frac{1}{9}D^2 - \cdots\cdots\right] x^2$$

$$= \frac{2}{3}\left[x^2 - \frac{2}{3}D(x^2) + \frac{1}{9}D^2(x^2)\right]$$

$$= \frac{2}{3}\left[x^2 - \frac{2}{3}(2x) + \frac{1}{9}(2)\right]$$

$$= \frac{2}{3}\left[x^2 - \frac{4}{3}x + \frac{2}{9}\right]$$

$$= \frac{2}{3}x^2 - \frac{8}{9}x + \frac{4}{27}$$

Hence, the general solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$= e^{-x}(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x) + \frac{2}{3}x^2 - \frac{8}{9}x + \frac{4}{27}$$

## Exercise-10

Solve the following differential equations :

* (1) $(D^3 - D^2 - 6D)y = x^2 + 1$ (G.T.U. June 2013) [07]

**Ans.** : $y = c_1 + c_2 e^{3x} + c_3 e^{-2x} - \frac{1}{6}\left(\frac{x^3}{3} - \frac{x^2}{6} + \frac{25}{18}x\right)$

(2) $(D^2 + 2D + 1)y = 2x + x^2$

**Ans.** : $y = (c_1 + c_2 x)e^{-x} + x^2 - 2x + 2$

Case IV : when $R(x) = e^{ax} \cdot V(x)$ where $V(x)$ is a fn. of $x$

$$P.I. = \frac{1}{f(D)} R(x)$$

i.e $V(x) = \sin(ax+b)$ or

$= \cos(ax+b)$ or

$= x^m$

$$= \frac{1}{f(D)} e^{ax} \cdot V(x)$$

$$= e^{ax} \frac{1}{f(D+a)} V(x)$$

Now proceed as before ( case II and case III )

**Case V : When $R(x) = x^m V(x)$ where $m \in N$**

(1) **For $m = 1$ :**

$$P.I. = \frac{1}{f(D)} R(x)$$

$$= \frac{1}{f(D)} x V(x)$$

$$= x \frac{1}{f(D)} V(x) - \frac{f'(D)}{[f(D)]^2} V(x)$$

Now proceed as before.

(2)

(2) **For $m \neq 1$ :**

We know that $e^{iax} = \cos ax + i \sin ax$

$\therefore \cos ax =$ Real part of $e^{iax}$

$\sin ax =$ Imaginary part of $e^{iax}$

(i) $P.I. = \frac{1}{f(D)} R(x)$

$$= \frac{1}{f(D)} x^m \cos ax$$

$$= \text{Real part of } \frac{1}{f(D)} x^m e^{iax}$$

(ii) $P.I. = \frac{1}{f(D)} R(x)$

$$= \frac{1}{f(D)} x^m \sin ax$$

$$= \text{Imaginary part of } \frac{1}{f(D)} x^m e^{iax}$$

Now proceed as before.