Gliders in cellular automata on Penrose tilings

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In this paper, we present the first glider capable of navigating an aperiodic tiling. It inhabits a four-state outer-totalistic cellular automaton, and operates on generic tilings of quadrilaterals. We investigate its behaviour on both the P2 (kite and dart) and P3 (rhombus) Penrose tilings, and characterise the different types of path it can follow. Further, we note that the path followed by the glider on the P2 tiling is a fractal curve generated by a simple Lindenmayer system, and compute its Hausdorff dimension.

Key words: Penrose tiling, gliders, tessellations, fractals, L-systems

1 INTRODUCTION

A Penrose tiling is a non-periodic tessellation of the plane with congruent copies of two distinct quadrilaterals, known as *prototiles* [2]. There are two common choices for these prototiles, both of which were discovered by Roger Penrose [2]. The P3 tiling involves 'thin' and 'fat' rhombi; the prototiles of the P2 tiling are the 'kite' and 'dart'. Both tilings are *mutually locally derivable* from each other, which means that they possess the same symmetry and statistical behaviour. However, the differences at the lowest level mean that the cellular automata are qualitatively different.

The most detailed analysis of cellular automata on Penrose tilings was done by Nick Owens and Susan Stepney [1]. They applied the rules of Conway's Game of Life to both the P2 and P3 tilings and conducted a computer-assisted search of the oscillators they support. Unlike cellular automata on

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regular tilings, such as the original Game of Life, no gliders (mobile configurations) were discovered. In the Game of Life, the glider is among the most useful patterns involved in larger constructions; it can transmit information across an infinite background of quiescent ground cells, enabling the construction of more complicated patterns such as computers.

The glider in the original Game of Life translates and reflects itself every two generations, causing it to move in a diagonal line. The underlying translational symmetry of the grid means that it will continue *ad infinitum*, and not suddenly 'crash'. Conversely, the Penrose tiling has no translational symmetry, so doubts were cast on whether gliders could exist. Andrew Trevorrow conjectured [10] that it was impossible for a glider to exist on a Penrose tiling, offering a prize of 100 Australian dollars for a counter-example.

We expand on the work by Owens and Stepney [1], using the same 'generalised Moore neighbourhood'. The *neighbours* of a tile are defined to be those which share at least one vertex with that tile. Due to restrictions imposed by the simulating software, *Ready* [7], all neighbours must be treated equally. This complicates matters, making it more difficult to construct a glider.

2 GLIDERS ON THE P3 TILING

The P3 tiling consists of rhombi, which are by definition parallelograms. Suppose we can create a glider capable of entering a rhombus through one edge and exiting through the opposite (parallel) edge. If we rotate the tiling such that this edge is horizontal, the ordinate (y-coordinate) of the glider must either monotonically increase or decrease. Hence, it cannot become trapped in a closed loop. There are ten possible directions such a glider can move in, making it more flexible than the original glider in the Game of Life, which can only move in four directions (North-east, North-west, South-east and South-west).

These ribbons of rhombi (see Figure 1) are actually bounded by two parallel straight lines. These ribbons are artefacts of de Bruijn's *pentagrid* [11] [2] method of constructing a Penrose tiling. They correspond to projections of four-dimensional hyperplanes of the five- dimensional integer lattice $\mathbb{Z}^5$ onto the Euclidean plane $\mathbb{R}^2$. Every rhombus is uniquely defined as the intersection of two non-parallel ribbons, and the converse is also true: any two non-parallel ribbons intersect at a single rhombus.

Our glider inhabits a four-state cellular automaton. In its small phase, the glider comprises a head 1 and tail 2 sharing an edge; the remainder of the universe is in the ground state 0. After one generation, all cells adjacent to

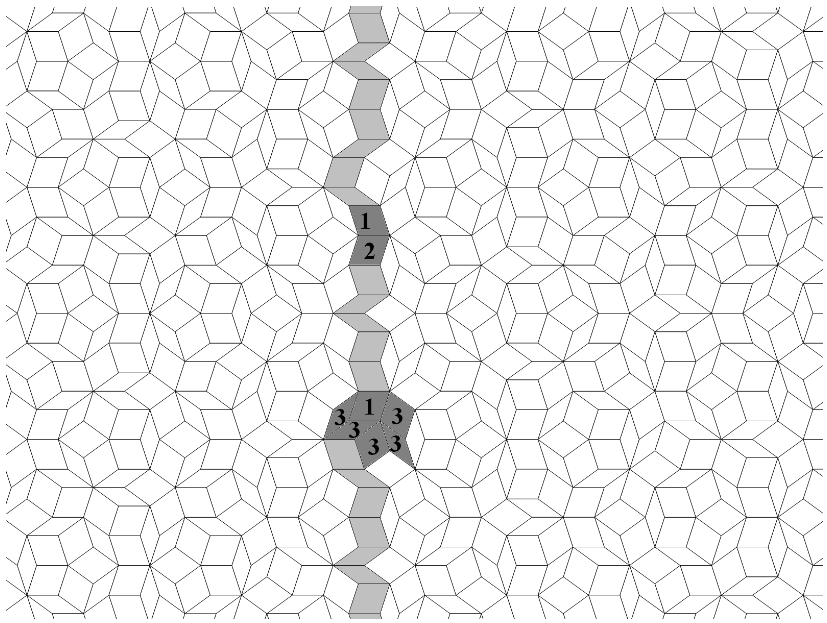

FIGURE 1
Two phases of a glider on the P3 Penrose tiling, showing the ‘ribbon’ of rhombi traversed by the glider. Unless otherwise specified, a tile is in the ground state 0.

Current state	Neighbour conditions			Next state
0	$n_1 \geq 1$	$n_2 \geq 1$	$*$	3
0	$n_1 \geq 1$	$*$	$n_3 \geq 2$	3
1	$*$	$*$	$n_3 \geq 1$	2
1	$*$	$*$	$*$	1
2	$*$	$*$	$*$	3
$*$	$*$	$*$	$*$	0

TABLE 1
The four-state cellular automaton. The next state of the cell is determined by the first rule that applies. n_i indicates the number of neighbours in state i. Where an asterisk (*) appears, it can stand for anything. The last transition, where all of the conditions are asterisks, is thus the default transition when no other rules apply.

both the head and tail, including the tail itself, become wing cells 3. They decay into the ground state during the next generation, whilst the nearby head changes into a tail and a new head is reborn ahead of the original one. The relevant rules are summarised concisely in Table 1.

Robert Munafo has sketched a proof [4] that the glider works on any tiling of quadrilaterals which is locally planar and has at least three quadrilaterals meeting at each vertex. For instance, it can travel across the order-5 square tiling of the hyperbolic plane. More interestingly, the cellular automaton can be explored on the P2 tiling of kites and darts.

3 LOOPING OSCILLATORS ON THE P2 TILING

The original proof that the glider does not return to its initial position breaks down on the kite and dart tiling, as the tiles are not parallelograms. Indeed, the dart is not even convex. The glider can actually loop, and it appears that this is the most common eventuality. The shortest loop is that of length 10, resulting in an oscillator of period 20 when populated with a single glider. As this is not technically a glider, as it does not travel arbitrarily far from the origin, we will instead refer to it as a *looper*. There is also a period-40 oscillator, where a looper orbits a larger decagon of tiles known as a *cartwheel* (see Figure 2).

There are also larger loops. For example, Andrew Trevorrow discovered [9] loopers of periods 160 and 1240 using the computer program *Ready*. It

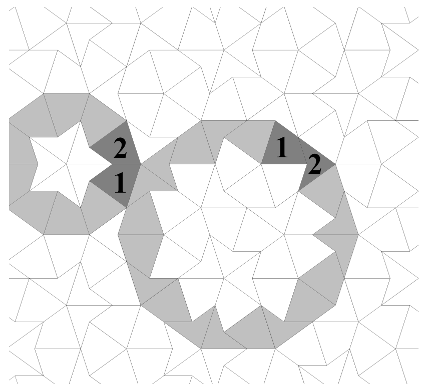

FIGURE 2
Oscillators of periods 20 and 40 constructed by placing a single looper on paths of length 10 and 20, respectively.

transpires that the loops can be classified into two infinite families, namely those with approximate decagonal symmetry and those with pentagonal symmetry. Some of these are shown in Figure 3.

It is convenient, at this point, to adopt a notation for describing loops. We use a lower-case k to indicate a kite where the looper leaves a short edge, and an upper-case K to indicate a kite where the looper leaves a long edge. Similarly, d and D are used to indicate darts. The loop of length 10 is represented by $kKkKkKkKkK$, whereas that of length 20 is represented by $dKdKkDdKdKkDkDdKkDkD$. Note that the letters alternate in case; for every tile, the short edges are opposite the long edges.

Geometrically, we obtain a larger area of a Penrose tiling by a process known as *deflation*, where the number of tiles is asymptotically multiplied by ϕ^2. Performing three successive deflations causes loops to be increased in size. This can be modelled by a simple *Lindenmayer system* [8] (or *L-system*):

- $k \to kKkDdKdK$
- $K \to kDkDdKkK$
- $d \to kK$
- $D \to kK$

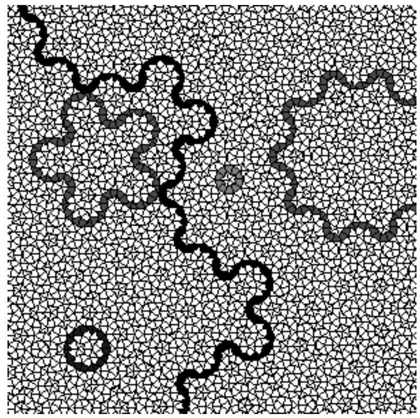

FIGURE 3
Loops of varying length. The longest loop in the diagram is at least period 1240, but may be part of one of the even larger loops, or could indeed be an infinite path.

By iterating this L-system in reverse, we can show that the only loops are those obtained by deflating the length-10 loop $kKkKkKkKkK$ and the length-20 loop $dKdKkDdKdKkDkDdKkDkD$. The resultant lengths of loops are summarised in Table 2.

Loopers can exist on regular tilings as well. Universal construction arguments guarantee that the Game of Life supports loopers of arbitrary length, but there are no explicit examples at the time of writing. In cellular automata with large neighbourhoods, however, such as *Larger than Life* [3], loopers have been discovered. In June 2001, Dean Hickerson discovered a period-552 looper in such a cellular automaton [6]. Dave Greene realised that the period can be reduced by placing multiple copies at regular intervals in the same loop, writing 'actually, 12 of these wondrous beasts can dance in a circle to create a period 46 oscillator'. This is shown in Figure 4.

Returning to the P2 Penrose tiling, the situation is more impressive. The periods of decagonal loopers form a sequence $40, 200, 1240, \ldots$. We can divide by 40 and prepend 0 as a zeroth term to obtain the related sequence $0, 1, 5, 31, \ldots$. This has the following linear recurrence relation, rather like the Fibonacci sequence:

$$P_{n+2} = 5P_{n+1} + 6P_n$$

pentagonal	decagonal
10	20
80	100
460	620
2780	3700
16660	22220
99980	133300
599860	799820
3599180	4798900
21595060	28793420
129570380	172760500
...	...

TABLE 2
Lengths of loops of each symmetry type. To calculate the fundamental period of the looper, multiply the path length by 2.

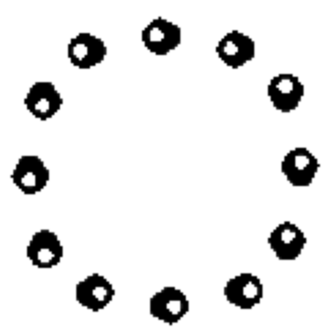

FIGURE 4
Dave Greene's oscillator of period 46, constructed by placing twelve copies of Dean Hickerson's looper at regular intervals. This exists in the *Larger than Life* cellular automaton with rule string 'R7,C0,M1,S65..114,B65..95,NM'.

Let r be a positive integer such that $\gcd(r, 6) = 1$, and consider the sequence modulo r. We can extrapolate the sequence in reverse as well as forwards. This means it must be periodic, so the initial term 0 occurs infinitely often. As such, for every r coprime with 6, there exist oscillators of period $40r$, and all (sufficiently large) factors thereof. Consequently, oscillators exist for every sufficiently large period not divisible by 3 or 16. This is not quite as good as the Game of Life, where all sufficiently large oscillator periods exist, but is the next best thing.

4 FROM LOOPERS TO GLIDERS

Recall that larger loops can be obtained from smaller ones by iteration of an L-system. We can actually iterate the L-system *ad infinitum* to the initial string k to result in an infinite path of kites and darts. By the Extension Theorem [5], this is part of a valid Penrose tiling. In other words, we have an actual glider, rather than merely a sequence of arbitrarily long loopers.

If a loop has m kites and n darts, we represent it with the column vector $\begin{pmatrix} m \\ n \end{pmatrix}$. After k successive iterations of the L-system, the number of kites and darts in the deflated loop is given by:

$$\begin{pmatrix} 5 & 2 \\ 3 & 0 \end{pmatrix}^k \begin{pmatrix} m \\ n \end{pmatrix}$$

The dominant eigenvalue of this matrix is 6, which means that there are asymptotically six times as many tiles in the loop after applying a single iteration of the L-system. By comparison, the diameter of the loop increases by merely a factor of ϕ^3. Hence, the path taken by the looper is a convoluted fractal curve with a Hausdorff dimension of $\frac{\log 6}{3 \log \phi}$. As such, the glider is slower than any glider on any periodic tiling.

As the P2 and P3 Penrose tilings are mutually locally derivable [2], one could theoretically design a multi-state rule which emulates a P3 cellular automaton on a P2 grid. As a consequence, the P2 kite and dart tiling does support linear-speed gliders, but they are probably more complex than their counterparts on the P3 rhombus tiling.

5 CONCLUSION

In conclusion, we have presented gliders in cellular automata on two aperiodic tilings, where none previously existed. It is possible that this could be used to engineer more complicated patterns, as was the case in Conway's Game of Life. The next step would be to choose a suitable chaotic rule supporting natural gliders and discover a 'gun' capable of periodically emitting these gliders.

We have also characterised the closed and open paths traced by the glider. This is not a feature of the cellular automaton; it is instead a fundamental aspect of the underlying geometry of the Penrose tiling.

6 ACKNOWLEDGEMENTS

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