1

21.6 #4

PROVE: If p_i and q_i are Cauchy, show $p_i \cdot q_i$ is ALSO, where $p_i, q_i \in \mathbb{Q}$ $\overset{?}{[\mathbb{Q} \text{ or } \mathbb{R}]}$

RECALL 1 (Exer 21.6 #5) If p_i is Cauchy, then p_i is bounded for some M_p

i.e. $|p_i| \leq M_p$

and $|q_i| \leq M_q$

RECALL 2 (Triangle) $|a+b| \leq |a| + |b|$ $a, b \in \mathbb{R}$

3 (Reverse? triangle) $\big|\, |a| - |b| \,\big| \leq |a-b|$ $a, b \in \mathbb{R}$

4 Suppose $x < M$ $x, M \in \mathbb{R}$, $M > 0$

skip

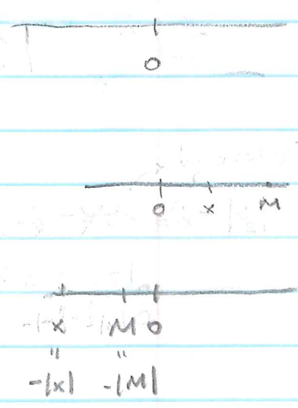
then $|x| < |M|$

Pf

CASE 1 $x > 0 \Rightarrow x = |x| \Rightarrow |x| < M$

CASE 2 $x < 0 \Rightarrow x = -|x| \Rightarrow -|x| < |M|$

?



NEED TO SHOW

$\forall \epsilon > 0$

$|p_i q_i - p_j q_j| < \epsilon$ $\forall i, j > M$, some M

Since q_i is bounded, $|q_i| \leq M_q$ some $M_q > 0$

Consider $\overline{p_i > 0}$ $p_i |q_i| \leq p_i M_q$

(1) so $|p_i q_i| \leq |p_i M_q|$

(2) Likewise $|p_j q_j| \leq |p_j M_q|$

SUBTR (1) − (2) REVERSE $\triangle$ (NOT EXACTLY)

(3) $|p_i q_i| - |p_j q_j| \leq |p_i M_q| - |p_j M_q| \leq |p_i - p_j||M_q|$

NEED TO SHOW $p_i < 0$ (NOT DONE) + HAVE PROBLEM

But we know p_i is Cauchy, so for some M_p

$$|p_i - p_j| < \epsilon \quad \forall i,j > M_p$$

choose ϵ / M_q as bound

$$\Rightarrow |p_i - p_j| < \epsilon / M_q \quad \forall i,j > M_p$$

SUBSTITUTE IN (3)

$$|p_i q_i| - |p_j q_j| \leq \epsilon \quad \forall i,j > M_p$$

$\Rightarrow$ $p_i q_i$ is Cauchy