11/14/2018

Exercise 17.2:

Recursive definition: $\quad sum(0) = 0 \quad (1)$

$\quad sum(n+1) = sum(n) + (n+1)^2 \quad (2)$

Proposition: For every natural number n,

$$sum(n) = \frac{1}{6} n (n+1)(2n+1) \quad (3)$$

Base case: Replacing n by 0 in (3) results in $sum(0)=0$ (1), as required.

Induction step:

Take (3) as the inductive hypothesis. We must show that if (3) holds for some particular n, then it also holds for n replaced by $n+1$:

$$
\begin{aligned}
sum(n+1) &= sum(n) + (n+1)^2 \text{, by (2)} \\
&= \frac{1}{6} n(n+1)(2n+1) + (n+1)^2 \text{, by hypothesis} \\
&= \frac{1}{6}(n+1)(2n^2+n) + \cancel{(n^2+2n+1)}\,(n+1)(n+1) \\
&= \frac{1}{6}(n+1)(2n^2+n) + \frac{1}{6}(n+1)(6)(n+1) \\
&= \frac{1}{6}(n+1)((2n^2+n) + 6(n+1)) \\
&= \frac{1}{6}(n+1)((2n^2+n) + 6n+6) \\
&= \frac{1}{6}(n+1)(2n^2+7n+6) \\
&= \frac{1}{6}(n+1)(n+2)(2n+3) \\
&= \frac{1}{6}[n+1]([n+1]+1)(2[n+1]+1),
\end{aligned}
$$

as required

Taken together, the proofs of the base case and the induction step establish the proposition.