Logic and Proof Chapter 20 Exercises

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February 10, 2019

1. Suppose that, at a party, every two people either know each other or don't. In other words, "x knows y" is symmetric. Also, let us ignore the complex question of whether we always know ourselves by restricting attention to the relation between distinct people; in other words, for this problem, take "x knows y" to be irreflexive as well.

   Use the pigeonhole principle (and an additional insight) to show that there must be two people who know exactly the same number of people.

   Proof. Use n to represent the number of people at the party. Then for each person, they must know between 0 and $n-1$ other people at the party.

   Either there is someone at the party who knows no other people, or there isn't.

   Suppose there is. Then there can't be anyone who knows everybody, because knowing each other is mutual, so the person who knows nobody would have to know them. And if there's anyone else at the party who knows zero people, then we've shown that there are two people who know the same number of people, so we're done. So the only other possibility to is that the remaining $n-1$ people each know somewhere between 1 and $n-2$ other people. By the pigeonhole principle, two of those people must know the same number of people.

   On the other hand, suppose there is nobody at the party who knows nobody. Then the $n-1$ people must know between 1 and $n-1$ people. That's a function from a set of cardinality $n-1$ to $n-2$, so again the pigeonhole principle tells us that two of those people must know the same number of people.

2. Show that in any set of $n+1$ integers, two of them are equivalent modulo n.

Proof. Any set of $n+1$ integers has cardinality $n+1$, so it has a bijection with $[n+1]$. Call the set S and choose an indexing bijection $i : [n+1] \to S$. Then define the function $m(s) = s \bmod n$ from S to $[n]$. Then $m \circ i$ is a function from $[n+1]$ to $[n]$. By the pigeonhole principle, it cannot be injective. Thus there exist natural numbers a and b in $[n+1]$ that are mapped to the same value by $m \circ i$. Then $i(a)$ and $i(b)$ are distinct elements of S that are equivalent modulo n.

3. Spell out in detail a proof of the second counting principle in Section 20.2: Let A and B be finite sets. Then $|A \times B| = |A| \cdot |B|$.

   Proof. Suppose $f : [m] \to A$ and $g : [n] \to B$ are bijections. Then we can construct a bijection $h : [m \cdot n] \to A \times B$, defined as follows. For $i \in [m \cdot n]$, by the quotient-remainder theorem, we have a unique q and r such that $0 \le r < m$ and $qm + r = i$. Then $h(i) = (f(r), g(q))$.

   To show that h is bijective, we will show that it has an inverse, $h^{-1} : A \times B \to [m \cdot n]$. Since f and g are bijective, they have inverses, f^{-1} and g^{-1}. We claim

$$h^{-1}((x, y)) = f^{-1}(x) \cdot m + g^{-1}(y).$$

   To verify that h^{-1} is a left inverse of h, we calculate

$$\begin{aligned} h^{-1}(h(i)) &= h^{-1}(f(r), g(q)) \text{ where } qm + r = i \wedge 0 \le r < m \\ &= f^{-1}(f(r)) \cdot m + g^{-1}(g(q)) \\ &= r \cdot m + q \\ &= i. \end{aligned}$$

   To verify that h^{-1} is a right inverse of h, we calculate

$$\begin{aligned} h(h^{-1}((x, y))) &= h(f^{-1}(x) \cdot m + g^{-1}(y)) \\ &= (f(r), g(q)) \end{aligned}$$

   where $q \cdot m + r = f^{-1}(x) \cdot m + g^{-1}(y)$ and $0 \le r < m$. But note that because g^{-1} has a codomain of $[m]$, the quotient-remainder theorem tells us that $q = f^{-1}(x)$ and $r = g^{-1}(y)$. Therefore

$$\begin{aligned} h(h^{-1}((x, y))) &= (f(r), g(q)) \\ &= (f(f^{-1}(x)), g(g^{-1}(y))) \\ &= ((x, y)). \end{aligned}$$

   Since $h : [m \cdot n] \to A \times B$ is a bijection, $|A \times B| = |A| \cdot |B|$.

4. An ice cream parlor has 31 flavors of ice cream.

   a. Determine how many three-flavor ice-cream cones are possible, if we care about the order and repetitions are allowed. (So choosing chocolate-chocolate-vanilla scoops, from bottom to top, is different from choosing chocolate-vanilla-chocolate.)
   Ans. $31^3 = 29791$.

   b. Determine how many three flavor ice-cream cones are possible, if we care about the order, but repetitions are not allowed.
   Ans. $P(31, 3) = \frac{31!}{(31-3)!} = 31 \cdot 30 \cdot 29 = 26970$.

   c. Determine how many three flavor ice-cream cones are possible, if we don't care about the order, but repetitions are not allowed.
   Ans. $C(31, 3) = \frac{31!}{3!(31-3)!} = \frac{31 \cdot 30 \cdot 29}{3 \cdot 2 \cdot 1} = \frac{26970}{6} = 4495$.

5. A club of 10 people has to elect a president, vice president, and secretary. How many possibilities are there:

   a. if no person can hold more than one office?
   Ans. $P(10, 3) = 10 \cdot 9 \cdot 8 = 720$.

   b. if anyone can hold any number of those offices?
   Ans. $10^3 = 1000$.

   c. if anyone can hold up to two offices?
   Ans. This is the same as above, except we need to subtract the cases where all three offices are occupied by the same person. There are 10 people, so 10 ways that could happen. Thus the answer is $1000 - 10 = 990$.

   d. if the president cannot hold another office, but the vice president and secretary may or may not be the same person?
   Ans. This gives us 10 choices for president. For each chosen president, there are 9 choices each for vice president and secretary. Thus the total number of possibilities is $10 \cdot 9 \cdot 9 = 810$.

6. How many 7 digit phone numbers are there, if any 7 digits can be used?

   Ans. $10^7 = 10000000$.

   How many are there if the first digit cannot be 0?

   Ans. $9 \cdot 10^6 = 9000000$.

7. In a class of 20 kindergarten students, two are twins. How many ways are there of lining up the students, so that the twins are standing together?

   Ans. There are 19 slots in which the twins can be standing next to each other. In each of those slots, the twins could be in one of two orders. There are 18 remaining students, who can be arranged in any of 18! orders. So the total number of ways of lining up the students with the twins standing together is $19 \cdot 18! \cdot 2$, or 243,290,200,817,664,000.

8. A woman has 8 murder mysteries sitting on her shelf, and wants to take three of them on a vacation. How many ways can she do this?

   Ans. $C(8,3) = 8!/(8-3)!3! = (8 \cdot 7 \cdot 6)/(3 \cdot 2 \cdot 1) = 56$.

9. In poker, a "full house" is a hand with three of one rank and two of another (for example, three kings and two fives). Determine the number of full houses in poker.

   Ans. Choose one of thirteen ranks to be the one with the three of a kind. To choose the three out of four cards, there are four options. Choose one of the remaining twelve ranks to be the one with the pair. To choose the two out of four cards, there are $(4 \cdot 3)/2 = 6$ ways. Thus the total number of full houses is $13 \cdot 12 \cdot 4 \cdot 6 = 3744$.

10. We saw in Section 20.4 that

$$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}.$$

   Replacing $k+1$ by k, whenever $1 \le k \le n$, we have

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$$

   Use this to show, by induction on n, that for every $k \le n$, that if S is any set of n elements, $\binom{n}{k}$ is the number of subsets of S with k elements.

   Proof. For the base case, where $n = 0$, S must be the empty set, and there is exactly one subset of it with 0 elements: that is, the empty set itself. This agrees with the formula $\binom{0}{0} = 1$.

   Our inductive hypothesis is that, for any n, for every $k \le n$, that if S is a set of n elements, $\binom{n}{k}$ is the number of subsets of S with k elements.

From this we must prove that for any set with $n+1$ elements, and $k \leq n+1$, that $\binom{n+1}{k}$ is the number of subsets of S with k elements.

Consider the case where $k = 0$. Since there is only one subset of S with 0 elements (the empty set), the answer should be 1, and indeed, $\binom{n+1}{0} = \frac{(n+1)!}{(n+1)! \cdot 1!} = 1$.

In the case where $k = n+1$, there can only be one subset of S, which is S. We have $\binom{n+1}{n+1} = \frac{n+1!}{0!(n+1)!} = 1$, so that works, too.

Finally, consider the case when $1 \leq k \leq n$. To figure out how many subsets of S (with $n+1$ elements) there are of size k, we enumerate the elements of S. We divide the subsets into two cases: ones that include the last element of S, and ones that don't.

The subsets that don't include the last element are the same as the subsets of size k of S with the last element removed. By the inductive hypothesis, that's $\binom{n}{k}$. The subsets that *do* include the last element also must include $k-1$ elements from the first n elements of S. Again by the inductive hypothesis, there are $\binom{n}{k-1}$ such subsets.

Adding these together, we get $\binom{n}{k} + \binom{n}{k-1}$, which as stated above equals $\binom{n+1}{k}$. Thus we have proven the inductive case.

11. How many distinct arrangements are there of the letters in the word MISSISSIPPI?

    (Hint: this is tricky. First, suppose all the S's, I's, and P's were painted different colors. Then determine how many distinct arrangements of the letters there would be. In the absence of distinguishing colors, determine how many times each configuration appeared in the first count, and divide by that number.)

    Ans. Suppose all the letters are painted different colors, so that each letter is distinct. Since there are 11 letters, there are 11! permutations, or 39,916,800.

    Now, how many ways can we rearrange each of the duplicate letters? There are four Ss, so they can be arranged 4! ways. There are four Is, so they can be arranged 4! ways. And there are two Ps, so they can be arranged 2! ways. This gives us a total of 4!4!2!, or 1,152 ways.

    That means the total number of ways that the letters in MISSISSIPPI can be rearranged, ignoring rearrangment of the duplicate letters, is 11!/(4!4!2!), or 34,650.

12. Prove the inclusion-exclusion principle.

Proof. The inclusion-exclusion principle[1] is:

$$\left|\bigcup_{i<n} A_i\right| = \sum_{\emptyset \neq I \subseteq [n]} (-1)^{|I|-1} \left|\bigcap_{i \in I} A_i\right|. \tag{1}$$

We will proceed by induction on n.

The base case, when $n = 0$, is simple. Since there are no natural numbers less than zero, the union on the left-hand side is empty, so its cardinality is zero. On the right-hand side, the only subset I of $[0]$ is the empty set, which is explicitly disallowed in the sum, so we are summing zero terms, which also produces zero.

For the inductive step, we can assume the principle holds true for n, and must prove that it holds true for $n + 1$. We propose that the left-hand side expands thusly:

$$\left|\bigcup_{i<n+1} A_i\right| = \left|\bigcup_{i<n} A_i\right| - \left|\bigcup_{i<n} A_i \cap A_n\right| + |A_n|. \tag{2}$$

Why? Any element that appears in one of the A_i sets for $i < n$ and also appears in A_n gets counted as part of each, so we need to subtract out the count of such elements.

Now let's start with the equality we're trying to prove, and work backward. We're trying to show that

$$\left|\bigcup_{i<n+1} A_i\right| = \sum_{\emptyset \neq I \subseteq [n+1]} (-1)^{|I|-1} \left|\bigcap_{i \in I} A_i\right|. \tag{3}$$

Let's expand the RHS of that. To simplify it, define the function

$$F(J) = (-1)^{|J|-1} \left|\bigcap_{i \in J} A_i\right|.$$

Then we can abbreviate the RHS of (3) as

$$\sum_{\emptyset \neq I \subseteq [n+1]} F(I). \tag{4}$$

[1] The exponent on -1 has been corrected from the text. Thanks to Ed Snow for pointing this out, along with providing the proof.

Consider that we can partition the subsets of $[n+1]$ into two sets: Those that contain n and those that don't. We can therefore rewrite (4) as

$$\sum_{\emptyset \neq I \subseteq [n]} F(I) + \sum_{I \subseteq [n]} F(I \cup \{n\})$$

and that can be further expanded to

$$\left(\sum_{\emptyset \neq I \subseteq [n]} F(I) \right) + \left(\sum_{\emptyset \neq I \subseteq [n]} F(I \cup \{n\}) \right) + F(\{n\}). \tag{5}$$

Our assertion is that the three terms in (5) correspond exactly, in order, to the terms on the RHS of (2).

Let's take each in turn. The first term on the RHS of (2) is equal to the first term in (5) by the inductive hypothesis:

$$\left| \bigcup_{i<n} A_i \right| = \sum_{\emptyset \neq I \subseteq [n]} (-1)^{|I|-1} \left| \bigcap_{i \in I} A_i \right| = \sum_{\emptyset \neq I \subseteq [n]} F(I).$$

The second term on the RHS of (2) can also be expanded via the inductive hypothesis, but with $A_i \cap A_n$ substituted in for A_i in the IH. This is safe to do because the IH is implicitly universally quantified over the A_i, so it's not really the *same* A that we're substituting in.

$$\begin{aligned} -\left| \bigcup_{i<n} A_i \cap A_n \right| &= -(-1)^{|I|-1} \left| \bigcap_{i \in I} A_i \cap A_n \right| \\ &= (-1)^{|I|} \left| \bigcap_{i \in I} A_i \cap A_n \right| \quad (6) \\ &= (-1)^{|I|} \left| \left(\bigcap_{i \in I} A_i \right) \cap A_n \right| \quad (7) \\ &= (-1)^{|I|} \left| \bigcap_{i \in I \cup \{n\}} A_i \right| \\ &= F(I \cup \{n\}). \end{aligned}$$

Note that we can proceed from (6) to (7) due to the fact that intersection distributes over intersection.

The third term on the RHS of (2) is simple to expand to the first term of (5):

$$|A_n| = \left| \bigcap_{i \in \{n\}} A_i \right| = F(\{n\}).$$

Since we have shown that each term in the RHS of (2) is correspondingly equal to each term in (5), we have proven our inductive step (3).

13. Use the inclusion-exclusion principle to determine the number of integers less than 100 that are divisible by 2, 3, or 5.

    There are $\lfloor 99/2 \rfloor + 1 = 50$ integers less than 100 divisible by 2. There are $\lfloor 99/3 \rfloor + 1 = 34$ integers less than 100 divisible by 2. There are $\lfloor 99/5 \rfloor + 1 = 20$ integers less than 100 divisible by 2.

    To be divisible by both 2 and 3, a number has to be divisible by 6. There are $\lfloor 99/6 \rfloor + 1 = 17$ of these. To be divisible by both 2 and 5, a number has to be divisible by 10. There are $\lfloor 99/10 \rfloor + 1 = 10$ of these. To be divisible by both 3 and 5, a number has to be divisible by 15. There are $\lfloor 99/15 \rfloor + 1 = 7$ of these.

    Finally, to be divisible by 2, 3, and 5, a number has to be divisible by 30. There are $\lfloor 99/30 \rfloor + 1 = 4$ of these.

    By the inclusion exclusion principle, there are $50 + 34 + 20 - 17 - 10 - 7 + 4 = 74$ integers less than 100 divisible by 2, 3, or 5.

14. Show that the number of *unordered* selections of k elements from an n-element set is $\binom{n+k-1}{k}$.

    Hint: consider $[n]$. We need to choose some number i_0 of 0's, some number i_1 of 1's, and so on, so that $i_0 + i_1 + \ldots + i_{n-1} = k$. Suppose we assign to each such tuple a the following binary sequence: we write down i_0 0's, then a 1, then i_1 0's, then a 1, then i_2 0's, and so on. The result is a binary sequence of length $n + k - 1$ with exactly k 1's, and such binary sequence arises from a unique tuple in this way.

    Proof. Since any set with n elements has a bijection with $[n]$, we can count the number of unordered selections of k elements, with repetitions, from $[n]$.

    We need to choose some number i_0 of 1s, some number i_1 of 1s, and so on, so that $i_+ i + 1 + \ldots + i_{n-1} = k$. Suppose we assign to each tuple a binary sequence: we write down i_0 0s, then a 1, then i_1 0s, then another 1, and so on. But following the i_{n-1} 0s, we do not write a 1 at the end. Thus, there are exactly k 0s and $n-1$ 1s in the sequence, for a total of $n+k-1$ binary digits.

Note that the placement of the 1s within the sequence partitions the 0s, and is exactly what we can vary to make a different selection from $[n]$. To count them, we just need to choose k out of $n+k-1$ elements, to be the 0s. Thus the count is $\binom{n+k-1}{k}$, as required.

We have transformed the problem of choosing k elements from a set of n elements, with repetitions, into a problem of choosing k elements from a set of $n+k-1$ elements, without repetitions.