What follows is a distillation of Smullyan's argument, one owing a great deal to Mark Dominus.[7] Let *$* denote an arbitrary text string (hereinafter "string"), and let there be a printer that prints strings. Some of the strings that the printer might or might not print look like the following:

P**$* (meaning that the printer prints *$*)
NP**$* (meaning that the printer never prints *$*)
PR*$ (meaning that the printer prints *$$*)
NPR**$* (meaning that the printer never prints *$$*)

Note that NP**$$* and NPR**$* have the same meaning.

Consider the string NPR*NPR*, which asserts that the printer never prints NPR*NPR*. Now, either the printer prints NPR*NPR*, or it never does so. If the printer prints NPR*NPR*, it has printed a false string. But if the printer never prints NPR*NPR*, then NPR*NPR* is a true string that the printer never prints.

So either the printer sometimes prints false strings, or there are true strings that it never prints. Hence any printer that prints only true strings must fail to print some true strings. Conversely, any machine that prints every possible true string must print some false strings too.

Smullyan (1994) rightly drew attention to Tarski's theorem (which says that a finitely axiomatized formal system strong enough for Robinson arithmetic cannot define its own truth predicate), because its philosophical implications rival those of Gödel's theorem, yet is much easier to prove. First order Peano arithmetic is but one of many incompletable formal systems for which Tarski's theorem holds.

A.16. Robbins Algebras Are Boolean Algebras: McCune's Proof Restated in Boundary Notation

A *Robbins algebra* is a $\langle B, \bullet, ^{-} \rangle$ algebra of type $\langle 2,1 \rangle$. To obtain a boundary notation for Robbins algebra, simply erase $\bullet$ and enclose in parentheses that which lies under an overbar. In the case of a single letter, write $a' =_{\mathbf{df}} (a) \Leftrightarrow \bar{a}$, as is standard throughout this book. $\bullet$ commutes and associates by assumption. The sole additional postulate is the *Robbins equation* R, the dual of C6, the Huntington (1933) equation. For a Robbins algebra derivation of the Huntington equation $(a'b')(ab') = b$, see Mann (2003: §5).

In 1933, Robbins conjectured that his eponymous algebras were in fact Boolean. This conjecture was an open problem until McCune (1997: §2) proved it using computer-intensive methods. Dahn (1998) reworked McCune's proof to bring it closer to the canonical "tree of lemmas" style of mainstream mathematics. Below I rework Dahn's

7. http://blog.plover.com/math/Gdl-Smullyan.html .

proof into the style of demonstration employed in this book, as an example of how boundary notation can streamline nontrivial contemporary mathematics. The following demonstration invokes Dahn's variant of R, $((a'b)(ab)) = b$, 16 times.

Theorem. Robbins algebras are Boolean algebras.

Preliminaries. Let $k, l, m, n \in \mathrm{N}$. U0-U2 define Dahn's subscript notation. D1-D3 set out some preliminary facts. "TR" permits reordering concatenated subformulae.

U0: $a_0 =_{\mathbf{df}} (a'a)$

U1: $a_n =_{\mathbf{df}} a_{n-1}a = a_0\,\overbrace{a\ldots a}$

$= \overbrace{a\ldots a}\,a_0$

U2: $m+n = k+l \rightarrow a_m a_n = a_k a_l$.

D1: $a' = b' \rightarrow (a'c)$ [R] $= ((a(\underline{a'c}))(a'(\underline{a'c})))$ [b'/a']

$= ((a(\mathbf{b'}c))(a'(\mathbf{b'}c)))$ [R] $= (b'c)$.

D2: $((aa)a_0)$ [TR; U0] $= ((\mathbf{a'a})(aa))$ [R] $= a$.

D3: $(a_2'a)$ [TR; U1] $= (\underline{a}(\mathbf{aaa_0}))$ [D2] $=$

$(((\mathbf{aa})\mathbf{a_0})(aaa_0))$ [R] $= a_0$.

Remark. U0-U2 establish that numerical subscripts work like the power notation of the algebra we all learned at school, the algebra of the real field. This subscript notation greatly shortens expressions until we prove that J1 and C1 hold in Robbins algebra, in which case $a_n=a$. (J1 and C1, of course, follow immediately from the theorem we are proving!) Recall (§A.5) that if '=' is a congruence relation, then $a=b \rightarrow a'=b'$ and $ac= bc$. Hence D1 asserts, in effect, that '=' is a congruence relation. D2 and D3 do duty for the familiar Boolean identities $a''=a$, $aa=a$, and $\bot a=a$.

Dahn's proof makes heavy use of the Boolean function $\delta(a,b) =_{\mathbf{df}} \overline{\overline{a}\cup b}$, whose sentential logic equivalent is $a\,|\,b$. (Recall that $\{\,|\,,\neg\}$ is expressively adequate; see Table 3-5.) Since $\delta(a,b) \Leftrightarrow (a'b)$, then $\delta(a,b') = \delta(b,a')$, R $\Leftrightarrow \delta(a'b,(ab)) = b$, U0 $\Leftrightarrow \delta(a,a) = a_0$, D1 $\Leftrightarrow a' = b' \rightarrow \delta(a,c) = \delta(b,c)$, D2 $\Leftrightarrow \delta(aa,a_0) = a$, and D3 $\Leftrightarrow \delta(a_2,a) = a_0$. Hence boundary notation makes the δ function less attractive.

DEM. Let Ln denote "Lemma n". Dahn then proves the lemmas L1-L7 below. Each lemma equates the first and last formula in its proof. I have replaced Dahn's α with χ because α is easily confused with a.

L1. $(a_3'a_0)$ [D3] $= (a_3'(a_2'a))$ [U1] $= ((a_2a)(a_2'a))$ [R] $= a$.

L2a. a [R] $= (((a_3'a_0)a)(a_3'a_0a))$ [U1] $= (((a_3'a_0)a)(a_3'a_1))$ [L1; TR] $= ((a_3'a_1)(aa))$.

Definition. $\chi =_{\mathbf{df}} (a_3'a_1(aa))$ [TR] $= ((aa)a_1a_3')$.

L2b. (aa) [R] $= (((a_3'a_1)(aa))(a_3'a_1(aa)))$ [Def. χ] $= (((a_3'a_1)(aa))\chi)$ [L2a] $= (a\chi)$.

L2c. $(a\chi)$ [L2b] $= (aa)$ [D1, a_0/c] $\rightarrow ((a\chi)a_0) = ((aa)a_0)$ [D2] $= a$.

L2d. a_3' [R] = $((((aa)a_1)a_3')((aa)a_1a_3'))$ [Def. χ] = $((((aa)a_1)a_3')\chi)$ [U1] = $((((aa)a_1)(aaa_1))\chi)$ [R] = $(a_1\chi)$.

L2. a_3' [L2d] = $(a_1\chi)$ [D1, a/c] $\rightarrow$ $(a_3'a)$ = $((a_1\chi)a)$ [TR,2x] = $(a(\chi a_1))$ [U1] = $(a(\chi aa_0))$ [L2c] = $(((\chi a)a_0)(\chi aa_0))$ [R] = a_0.

L3. $((a_1a_3)a)$ [R] = $((a_1a_3)((a_3'a)(a_3a)))$ [L2] = $((a_1a_3)(a_0(a_3a)))$ [U2; U1] = $((a_4a_0)(a_0a_4'))$ [TR,2x] = $((a_4'a_0)(a_4a_0))$ [R] = a_0.

L4. $((a_1a_2)a)$ [TR; U2] = $(a(a_3a_0))$ [L1] = $((a_3'a_0)(a_3a_0))$ [R] = a_0.

L5. $(a_0(a_1a_3))$ [L4] = $(((a_1a_2)a)(a_1a_3))$ [U1] = $(((a_1a_2)a)(a_1a_2a))$ [R] = a.

Definition. $\beta =_{\mathbf{df}} ((a_1a_3)aa_3')$ [TR] = $(a_3'a(a_1a_3))$.

L6. $(a\beta)$ [L1; TR] = $((a_0a_3')\beta)$ [L3] = $((((a_1a_3)a)a_3')\beta)$ [R] = a_3'.

L7. $(a\beta)$ [L5] = $((a_0(a_1a_3))\beta)$ [L2] = $(((a_3'a)(a_1a_3))\beta)$ [R] = (a_1a_3).

Mann (2003: 7) restates Winker's (1992) proof that if any of conditions (1) through (4) below can be derived in Robbins algebra, then all Robbins algebras are Boolean:
1. B2 or C3;
2. $\exists x \in B$ such that $xx = x$;
3. $\exists x,y \in B$ such that $xy = y$;
4. $\exists x,y \in B$ such that $(xy) = y'$.

L6 and L7 imply $(a_1a_3) = a_3'$, an instance of (4). Since L6 and L7 hold in any Robbins algebra, all Robbins algebras are Boolean. ∎

Remark. Dahn's proof relied on (4), the most complex of Winker's five sufficient conditions for Robbins algebras to be Boolean. Mann (2003: §§5,6) restates, using conventional notation consistently applied, the proofs of Winker (1992), then shows that Robbins algebra satisfies (3). For a related proof of the Robbins conjecture using the notation of *alpha* existential graphs (here set out in §6.1), see Kauffman (2001a).