UNIT-5

ARITHMETIC PROGRESSIONS

One of the endlessly alluring aspects of mathematics is that its thorniest paradoxes have a way of blooming into beautiful theories

1. Find the sum of all natural numbers amongst first one thousand numbers which are neither divisible 2 nor by 5

Ans: Sum of all natural numbers in first 1000 integers which are not divisible by 2 i.e. sum of odd integers.

$1 + 3 + 5 + \ldots\ldots\ldots + 999$

$n = 500$

$S_{500} = (500)^2$

$= 2,50,000$

No's which are divisible by 5

$5 + 15 + 25 \ldots\ldots + 995$

$n = 100$

$S_n = \frac{100}{2}[5 + 995]$

$= 50 \times 1000 = 50000$

$\therefore$ Required sum $= 250000 - 50,000$

= 200000

2. The fourth term of an AP is 0. Prove that its 25[th] term is triple its 11[th] term.

Ans: $a_4 = 0$

$\Rightarrow a + 3d = 0$

T.P $a_{25} = 3\,(a_{11})$

$\Rightarrow a + 24d = 3\,(a + 10d)$

$\Rightarrow a + 24d = 3a + 30d$

RHS sub $a = -3d$

$-3d + 24d = 21d$

LHS $3a + 30d$

$-9d + 30d = 21d$

LHS = RHS

Hence proved

3. Find the 20[th] term from the end of the AP 3, 8, 13……..253.

Ans: 3, 8, 13 ………….. 253

Last term = 253

a_{20} from end

$= 1 - (n-1)d$

$253 - (20-1)\,5$

$253 - 95$

= 158

4. If the p^{th}, q^{th} & r^{th} term of an AP is x, y and z respectively, show that x(q-r) + y(r-p) + z(p-q) = 0

Ans: p^{th} term $\Rightarrow x = A + (p-1)D$
q^{th} term $\Rightarrow y = A + (q-1)D$
r^{th} term $\Rightarrow z = A + (r-1)D$
T.P $x(q-r) + y(r-p) + z(p-q) = 0$
$=\{A+(p-1)D\}(q-r) + \{A + (q-1)D\}(r-p) + \{A+(r-1)D\}(p-q)$
$A\{(q-r) + (r-p) + (p-q)\} + D\{(p-1)(q-r) + (r-1)(r-p) + (r-1)(p-q)\}$
$\Rightarrow A.0 + D\{p(q-r) + q(r-p) + r(p-q) - (q-r) - (r-p) - (p-q)\}$
$= A.0 + D.0 = 0.$
Hence proved

5. Find the sum of first 40 positive integers divisible by 6 also find the sum of first 20 positive integers divisible by 5 or 6.

Ans: No's which are divisible by 6 are
6, 12 ……………. 240.
$S_{40} = \frac{40}{2}[6 + 240]$
$= 20 \times 246$
$= 4920$
No's div by 5 or 6
30, 60 ………… 600
$\frac{20}{2}[30 + 600] = 10 \times 630$
$= 6300$

6. A man arranges to pay a debt of Rs.3600 in 40 monthly instalments which are in a AP. When 30 instalments are paid he dies leaving one third of the debt unpaid. Find the value of the first instalment.

Ans: Let the value of I instalment be x $\quad S_{40} = 3600.$
$\Rightarrow \frac{40}{2}[2a + 39d] = 3600$
$\Rightarrow 2a + 39d = 180 \quad - \quad 1$
$S_{30} = \frac{30}{2}[2a + 29d] = 2400$
$\Rightarrow 30a + 435d = 2400$
$\Rightarrow 2a + 29d = 160 \quad - \quad 2$

Solve 1 & 2 to get
$d = 2$ $a = 51$.
$\therefore$ I instalment = Rs.51.

7. Find the sum of all 3 digit numbers which leave remainder 3 when divided by 5.

Ans: 103, 108……….998

$a + (n-1)d = 998$

$\Rightarrow 103 + (n-1)5 = 998$

$\Rightarrow n = 180$

$S_{180} = \frac{180}{2}[103 + 998]$

$= 90 \times 1101$

$S_{180} = 99090$

8. Find the value of x if $2x + 1$, $x^2 + x + 1$, $3x^2 - 3x + 3$ are consecutive terms of an AP.

Ans: $a_2 - a_1 = a_3 - a_2$

$\Rightarrow x^2 + x + 1 - 2x - 1 = 3x^2 - 3x + 3 - x^2 - x - 1$

$x^2 - x = 2x^2 - 4x + 2$

$\Rightarrow x^2 - 3x + 2 = 0$

$\Rightarrow (x-1)(x-2) = 0$

$\Rightarrow x = 1$ or $x = 2$

9. Raghav buys a shop for Rs.1,20,000.He pays half the balance of the amount in cash and agrees to pay the balance in 12 annual instalments of Rs.5000 each. If the rate of interest is 12% and he pays with the instalment the interest due for the unpaid amount. Find the total cost of the shop.

Ans: Balance = Rs.60,000 in 12 instalment of Rs.5000 each.

Amount of I instalment $= 5000 + \frac{12}{100} 60{,}000$

II instalment = 5000 + (Interest on unpaid amount)

$= 5000 + 6600 \qquad \left[\frac{12}{100} \times 55000\right]$

$= 11600$

III instalment = 5000 + (Interest on unpaid amount of Rs.50,000)

$\therefore$ AP is 12200, 11600, 11000

D = is 600

Cost of shop = 60000 + [sum of 12 instalment]

$$= 60,000 + \frac{12}{2}[24,400\text{-}6600]$$

$= 1,66,800$

10. Prove that $a_{m+n} + a_{m-n} = 2a_m$

Ans: $a_{m+n} = a_1 + (m + n - 1)d$
$a_{m-n} = a_1 + (m - n - 1)d$
$a_m = a_1 + (m-1)d$
Add 1 & 2

$$\begin{aligned} a_{m+n} + a_{m-n} &= a_1+(m+n-1)d + a_1 + (m-n-1)d \\ &= 2a_1+(m+n+m-n-1-1)d \\ &= 2a_1+ 2(m-1)d \\ &= 2[a_1+ (m-1)d] \\ &= 2[a_1+ (m-1)d] \\ &= 2a_m. \quad \text{Hence proved.} \end{aligned}$$

11. If the roots of the equation $(b-c)x^2 +(c-a)x +(a-b) = 0$ are equal show that a, b, c are in AP.

Ans: Refer sum No.12 of Q.E.
If $(b-c)x^2 + (c-a)x + (a-b)x$ have equal root.
$B^2-4AC=0$.
Proceed as in sum No.13 of Q.E to get $c + a = 2b$
$\Rightarrow b - a = c - b$
$\Rightarrow$ a, b, c are in AP

12. Balls are arranged in rows to form an equilateral triangle .The first row consists of one ball, the second two balls and so on. If 669 more balls are added, then all the balls can be arranged in the shape of a square and each of its sides then contains 8 balls less than each side of the triangle. find the initial number of balls.

Ans: Let their be n balls in each side of the triangle
$\therefore$ No. of ball (in Δ) $= 1 + 2+ 3\ldots\ldots\ldots.= \frac{n(n+1)}{2}$
No. of balls in each side square = n-8
No. of balls in square $= (n-8)^2$
APQ $\frac{n(n+1)}{2} + 660 = (n-8)^2$
On solving
$n^2 + n + 1320 = 2(n^2 - 16n + 64)$
$n^2 - 33n - 1210 = 0$
$\Rightarrow (n-55)(n+22) = 0$
$n=-22$ (N.P)
$n=55$

$\therefore$ No. of balls = $\dfrac{n(n+1)}{2} = \dfrac{55x56}{2}$

= 1540

13. Find the sum of $(1-\frac{1}{n})+(1-\frac{2}{n})+(1-\frac{3}{n})$ ……. upto n terms.

Ans: $\left(1-\frac{1}{n}\right)+\left(1-\frac{2}{n}\right)$ - upto n terms

$\Rightarrow$[1+1+…….+n terms] – [$\frac{1}{n}+\frac{2}{n}$+….+ n terms]

n –[S_n up to n terms]

$S_n = \frac{n}{2}$[2a + (n-1)d] (d = $\frac{1}{n}$, a = $\frac{1}{n}$)

$= \frac{n}{2}\left[\frac{2}{n}+(n-1)\frac{1}{n}\right]$

$= \frac{n+1}{2}$ (on simplifying)

n - $\frac{n+1}{2}$ =

$= \frac{n-1}{2}$ Ans

14. If the following terms form a AP. Find the common difference & write the next 3 terms3, 3+ √2, 3+2√2, 3+3√2……….

Ans: d= $\sqrt{2}$ next three terms 3 + 4$\sqrt{\ }^2$, 3 + 5$\sqrt{\ }^2$, 3 + 6$\sqrt{\ }^2$……..

15. Find the sum of a+b, a-b, a-3b, …… to 22 terms.

Ans: a + b, a – b, a – 3b, up to 22 terms

d= a – b – a – b = 2b

$S_{22} = \frac{22}{2}$ [2(a+b)+21(-2b)]

11[2a + 2b – 42b]

= 22a – 440b Ans.

16. Write the next two terms √12, √27, √48, √75……………….

Ans: next two terms $\sqrt{108}$, $\sqrt{147}$ AP is $2\sqrt{3}$, $3\sqrt{3}$, $4\sqrt{3}$, $5\sqrt{3}$, $6\sqrt{3}$, $7\sqrt{3}$ ……

17. If the p^{th} term of an AP is q and the q^{th} term is p. P.T its n^{th} term is (p+q-n).

Ans: APQ

$a_p = q$

$a_q = p$

$a_n = ?$

$a + (p-1)\ d = q$

$a + (q-1)\ d = p$

$d[p - q] = q - p$ Sub $d = -1$ to get $\Rightarrow = -1 \Rightarrow a = q + p - 1$

$a_n = a + (n - 1)d$

$= a + (n - 1)d$

$= (q + p - 1) + (n - 1) - 1$

$a_n = (q + p - n)$

18. If $\frac{1}{x+2}, \frac{1}{x+3}, \frac{1}{x+5}$ are in AP find x.

Ans: $\frac{1}{x+2}, \frac{1}{x+3}, \frac{1}{x+5}$ are in AP find x.

$$\frac{1}{x+3} - \frac{1}{x+2} = \frac{1}{x+5} - \frac{1}{x+3}$$

$$\Rightarrow \frac{1}{x^2+5x+6} = \frac{2}{x^2+8x+15}$$

On solving we get $x = 1$

19. Find the middle term of the AP 1, 8, 15….505.

Ans: Middle terms

a + (n-1)d = 505

a + (n-1)7 = 505

$n - 1 = \frac{504}{7}$

n = 73

$\therefore$ 37th term is middle term

a_{37} = a + 36d

= 1 + 36(7)

= 1 + 252

= 253

20. Find the common difference of an AP whose first term is 100 and sum of whose first 6 terms is 5 times the sum of next 6 terms.

Ans: a = 100

APQ $a_1 + a_2 + \ldots\ldots a_6 = 5\ (a_7 + \ldots\ldots + a_{12})$

$$6\left(\frac{a_1+a_6}{2}\right) = 5 \text{ x } 6 \left(\frac{a_7+a_{12}}{2}\right)$$

$\Rightarrow$ a + a + 5d = 5[a + 6d + a + 11d]

$\Rightarrow$ 8a + 80d = 0 (a = 100)

$\Rightarrow$ d = - 10.

21. Find the sum of all natural no. between 101 & 304 which are divisible by 3 or 5. Find their sum.

Ans: No let 101 and 304, which are divisible by 3.

102, 105…………303 (68 terms)

No. which are divisible by 5 are 105, 110……300 (40 terms)

No. which are divisible by 15 (3 & 5) 105, 120…… (14 terms)
$\therefore$ There are 94 terms between 101 & 304 divisible by 3 or 5. (68 + 40 – 14)
$\therefore S_{68} + S_{40} - S_{14}$
= 19035

22. The ratio of the sum of first n terms of two AP's is 7n+1:4n+27. Find the ratio of their 11^{th} terms .

Ans: Let a_1, a_2… and d_1, d_2 be the I terms are Cd's of two AP's.

$$\frac{S_n \text{ of one AP}}{S_n \text{ of II AP}} = \frac{7n+1}{4n+27}$$

$$\frac{\frac{m}{2}[2a_1+(n-1)d_1]}{\frac{m}{2}[2a_2+(n-1)d_2]} = \frac{7n+1}{4n+27}$$

$$\Rightarrow \frac{2a_1+(n-1)d_1}{2a_2+(n-1)d_2} = \frac{7n+1}{4n+27}$$

We have sub. n = 21.

$$\frac{2a_1+20d_1}{2a_2+20d_2} = \frac{7x21+1}{4(21)+27}$$

$$\Rightarrow \frac{a_1+10d_1}{a_2+10d_2} = \frac{148}{111}$$

$$= \frac{4}{3}$$

$\therefore$ ratio of their 11^{th} terms = 4 :3.

23. If there are (2n+1)terms in an AP ,prove that the ratio of the sum of odd terms and the sum of even terms is (n+1):n

Ans: Let a, d be the I term & Cd of the AP.

$\therefore a_k = a + (k-1) d$

s_1 = sum to odd terms

$s_1 = a_1 + a_3 + \dots\dots\dots a_{2n+1}$

$$s_1 = \frac{n+1}{2}[a_1 + a_{2n+1}]$$

$$= \frac{n+1}{2}[2a_1 + 2nd]$$

$s_1 = (n+1)(a+nd)$

s_2 = sum to even terms

$s_2 = a_2 + a_4 + \dots\dots a_{2n}$

$s_2 = \frac{n}{2}[a_2 + a_{2n}]$

$= \frac{n}{2}$ [a + d + a + (2n – 1)d]

=n [a + nd]

$\therefore$ $s_1 : s_2 = \frac{(n+1)(a+nd)}{n(a+nd)}$

$= \frac{n+1}{n}$

24. Find the sum of all natural numbers amongst first one thousand numbers which are neither divisible 2 nor by 5

Ans: Sum of all natural numbers in first 1000 integers which are not divisible by 2 i.e. sum of odd integers.

1 + 3 + 5 + ………. + 999

n = 500

$S_{500} = (500)^2$

= 2,50,000

No's which are divisible by 5

5 + 15 + 25 …….. + 995

n = 100

$S_n = \frac{100}{2}[5 + 995]$

= 50 x 1000 = 50000

$\therefore$ Required sum = 250000 – 50,000

= 200000

Question-1

Find the sum of the following A.P. 1, 3, 5, 7, ...,199.

Solution:

Given, $a = 1$, $d = 2$, $a_n = l = 199$,

$a + (n - 1) d = 199$

$1 + (n - 1) 2 = 199$

$\Rightarrow 1 + 2n - 2 = 199$

$\Rightarrow 2n = 200$

$\therefore \quad n = \frac{200}{2}$

$n = 100.$

$S_n = n/2 (a + l)$

$= 50(1 + 199)$

$= 50(200)$

$= 10000$

Question-2

Find the A.P. whose 10th term is 5 and 18th term is 77.

Solution:

given, 10th term of an A.P= 5

Þ $a + (10 - 1) d = 5$

$\Rightarrow a + 9d = 5$ ....(i)

and 18th term = 77

Þ$a + (18 - 1) d = 77$

$\Rightarrow a + 17d = 77$ ....(ii)

(ii) – (i), $8d = 72$

$\therefore d = 9$

Substituting the value of d = 9 in (i),

$a + 81 = 5$

$a = 5 - 81 = -76$

$\therefore$ The A.P. is – 76, - 67, ...

Question-3

In a certain A.P the 24th term is twice the 10th term. Prove that the 72nd term is twice the 34th term.

Solution:

Given, $a_{24} = 2a_{10}$

$a_{24} = a + 23d$ and $a_{10} = a + 9d$

To prove: $a_{72} = 2a_{34}$

$a_{72} = a + 71d$

$a_{34} = a + 33d$

$a_{24} = 2a_{10}$ (Given)

$a + 23d = 2(a + 9d)$

$a + 23d = 2a + 18d$

$a - 5d = 0$

$a = 5d$..........................(i)

$a_{72} = 2a_{34}$

$a + 71d = 2(a + 33d)$

$a + 71d = 2a + 66d$

$a - 5d = 0$

$a = 5d$..........................(ii)

from, (1) and (2) $a_{72} = 2a_{34}$

Hence proved.

Question-4

a, b and c are in A.P. Prove that b + c, c + a and a + b are in A.P.

Solution:

Given, a, b and c are in A.P.

$\therefore \quad b - a = c - b$

To prove : **b + c, c + a and a + b are in A.P.**

$c + a - (b + c) = a + b - (c + a)$

$\Rightarrow \; c + a - b - c = a + b - c - a$

$a - b = b - c$

$\Rightarrow b - a = c - b$

$\therefore$ a, b, c are in A.P.

$\therefore$ b + c, c + a and a + b are in A.P.

Question-5

If 9^{th} term of an A.P. is zero, prove that its 29^{th} term is double the 19^{th} term.

Solution:

9^{th} term = 0

$a_1 + 8d = 0$

$a_{29} = a_1 + 28d = a_1 + 8d + 20d = 0 + 20d = 20d$

$a_{19} = a_1 + 18d = a_1 + 8d + 10d = 0 + 10d = 10d$

$a_{29} = 2a_{19}$.

Question-6

Determine the A.P whose third term is 16 and the difference of 5^{th} from 7^{th} term is 12.

Solution:

Let the A.P. be a, a + d, a + 2d, ……..

$\Rightarrow$ The third term = $a_3 = a + 2d = 16$ ………………(i)

and seventh term = $a_7 = a + 6d$

Given that $a_7 - a_5 = 12$

$\Rightarrow (a + 6d) - (a + 4d) = 12$

$\Rightarrow a + 6d - a - 4d = 12$

$\Rightarrow 2d = 12$

$\Rightarrow d = 6$

Substituting the value of d = 6 in (i),

$a + 12 = 16$

$a = 4$

$\therefore$ The first term of the A.P. is 4 and the common difference is 6.

$\therefore$ The A.P. is 4, 10, 16, 22, 28, 34, …

$\therefore$ The fifth term = $a_5 = a + 4d$.

Question-7

The sum of the first six terms of an A.P is zero and the fourth term is 2. Find the sum of its first 30 terms.

Solution:

Let the sum of first 30 terms be S_{30}, first term be a, fourth term be a_4 and the sum of first six terms be S_6.

Given that $S_6 = 0$ and fourth term $a_4 = 2$

$\Rightarrow a + 3d = 2$ ....(i)

$S_6 = 0$

$\frac{n}{2}(2a + 5d) = 0$

$\Rightarrow 2a + 5d = 0$ ....(ii)

(i) × 2,

$2a + 6d = 4$ ....(iii)

(iii) – (ii),

$\therefore d = 4$

Substituting the value of d = 4 in (i),

$a + 3 \times (4) = 2$

$\Rightarrow a = 2 - 12 = -10$

$\therefore a_{30} = a + 29d$

$= -10 + 29 \times (4)$

$= -10 + 116$

$= 106$

$\therefore$ Sum to first 30 terms $= S_{30} = \frac{n}{2}(a + l)$

$= \frac{30}{2}(-10 + 106)$

$= 15 \times 96$

$= 1140.$

Question-8

An A.P consists of 60 terms. If the first and the last terms be 7 and 125 respectively, find 32nd term.

Solution:

Given, $n = 60$, $a_1 = 7$,

and $a_{60} = 125$

$\Rightarrow a_1 + 59d = 125$

$7 + 59d = 125$

$59d = 118$

$d = 118/59 = 2$

$a_{32} = a_1 + 31d = 7 + 31(2) = 7 + 62$

$\therefore a_{32} = 69.$

Question-9

Find the sum of the series 51 + 50 + 49 + ... + 21.

Solution:

$51 + 50 + 49 + ... + 21$

$a = 51$, $d = -1$, $a_n = 21$

$\therefore a + (n - 1) d = a_n$

$51 + (n - 1)(-1) = 21$

$(n - 1)(-1) = 21 - 51$

$n - 1 = 30$

$\therefore n = 31$

$\therefore$ Sum of the series $= \frac{31}{2}(51 + 21)$

$= \frac{31}{2} \times 72$

$= 1116$

$\therefore$ The sum of the series $51 + 50 + 49 + ... + 21 = 1116$.

Question-10

Three numbers are in A.P. If the sum of these numbers be 27 and the product 648, find the numbers.

Solution:

Let the three numbers be $a - d, a, a + d$.

$a - d + a + a + d = 27$

$3a = 27$

$a = 9$

$(a - d)(a)(a + d) = 648$

$a(a^2 - d^2) = 648$

$9(9^2 - d^2) = 648$

$9^2 - d^2 = 72$

$d^2 = 81 - 72$

$d^2 = 9$

$d = 3$

The numbers are 6, 9, 12.

Question-11

How many terms of A.P -10, -7, -4, -1, ... must be added to get the sum -104?

Solution:

-10, -7, -4, -1, ...

$a = -10, d = 3$

$S_n = \frac{1}{2}n\{2a + (n - 1) d\}$

$-104 = \frac{1}{2}n \{2(-10) + (n - 1) 3\}$

$= \frac{1}{2}n (-20 + 3n - 3)$

$-208 = n (3n - 23)$

$3n^2 - 23n + 208 = 0$

$3n^2 - 39n + 16n + 208 = 0$

$3n (n - 13) + 16 (n - 13) = 0$

$(n - 13) (3n + 16) = 0$

$\therefore n = 13$

$\therefore$ 13 terms must be added to get the sum of the A.P – 104.

Question-12

If the sum of p terms of an A.P is $3p^2 + 4p$, find its n^{th} term.

Solution:

$S_p = 3p^2 + 4p$

$t_n = S_n - S_{n-1}$

$= (3n^2 + 4n) - [3(n - 1)^2 + 4(n - 1)]$

$= (3n^2 + 4n) - [3(n^2 - 2n + 1) + 4(n - 1)]$

$= (3n^2 + 4n) - [3n^2 - 6n + 3 + 4n - 4]$

$= (3n^2 + 4n) - [3n^2 - 2n - 1]$

$= 3n^2 + 4n - 3n^2 + 2n + 1$

$= 6n + 1$

Therefore the n^{th} term is $6n + 1$.