For k=3, we describe a 4-chromatic UDG that fits within a disk of radius 1/Sqrt[3] + eps for arbitrarily small positive eps. Our construction is a bitquite elaborate and is perhaps the most significant result presented in this report.

We begin with a cycle of n vertices with n = 1 (mod 3) (and n>=7); we place the vertices at the corners of a regular polygon centered at the origin in such a way that the cycle winds around the origin (n-1)/3 times. It is easily seen that the distance of these vertices from the origin, which we denote r_1, asymptotically approaches 1/Sqrt[3] from above as n rises.

The remainder of our construction is repeated identically for all vertices of the cycle. Thus, for clarity, we describe the construction for one vertex, designated X, which we place at (0,-r_1), i.e. at "six o'clock" in polar-coordinate terms. We designate X's neighbors (in the cycle, not the polygon) as Y and Z, with Y being slightly anticlockwise of ten o'clock and Z being slightly clockwise of two o'clock. For a given choice of X, all the other vertices that we shall add will be in regions surrounding X, Y and Z whose diameter falls asymptotically to zero as n rises, thus delivering our desired result.

Next we add vertex A at unit distance from Z and 0.5 distance from the point that is the middle between Y and Z, oriented so that A is north-east of and close to Y.

Next we add vertex B at unit distance from Y and A, oriented so that B is close to Z.

We add edges ZA, AB and BY.

~~Next we add vertices A at unit distance from (and, of course, with edges joining it to) X and Z, and B at unit distance from X and Y, oriented so that A is close to Y and B is close to Z. Let the distance of A and B from the origin be r_2; clearly r_2 asymptotically approaches 1/Sqrt[3] from below as n rises.~~

~~Now we add vertices P_1,...P_j, j even, as a chain between Z and Y (i.e., P_i is at unit distance from P_i-1 for all I, P_0 is Z and P_j+1 is Y). The coordinates of the P_i are defined as follows:~~

~~- for I odd, P_i is close to Y, and for I even, P_i is close to Z~~

~~- for 1<=i<=j-1, if I = 0 or 1 (mod 4), P_i is r_1 from the origin, otherwise at r_2 from the origin~~

~~- P_j is at a distance r_3 from the origin such that r_1 >=r_3 >= r2.~~

~~As shown in Figure xxx, the upshot of this is that the P_i chain shuffles in an anticlockwise direction around the annulus defined by radii r_1 and r_2, in very much~~

~~the same manner as the odd cycle within a rectangle of height eps and length 2 described earlier.~~

~~Next we add vertices C_i, 0<=i<=j, connected to P_i and P_i+1 and near to X.~~

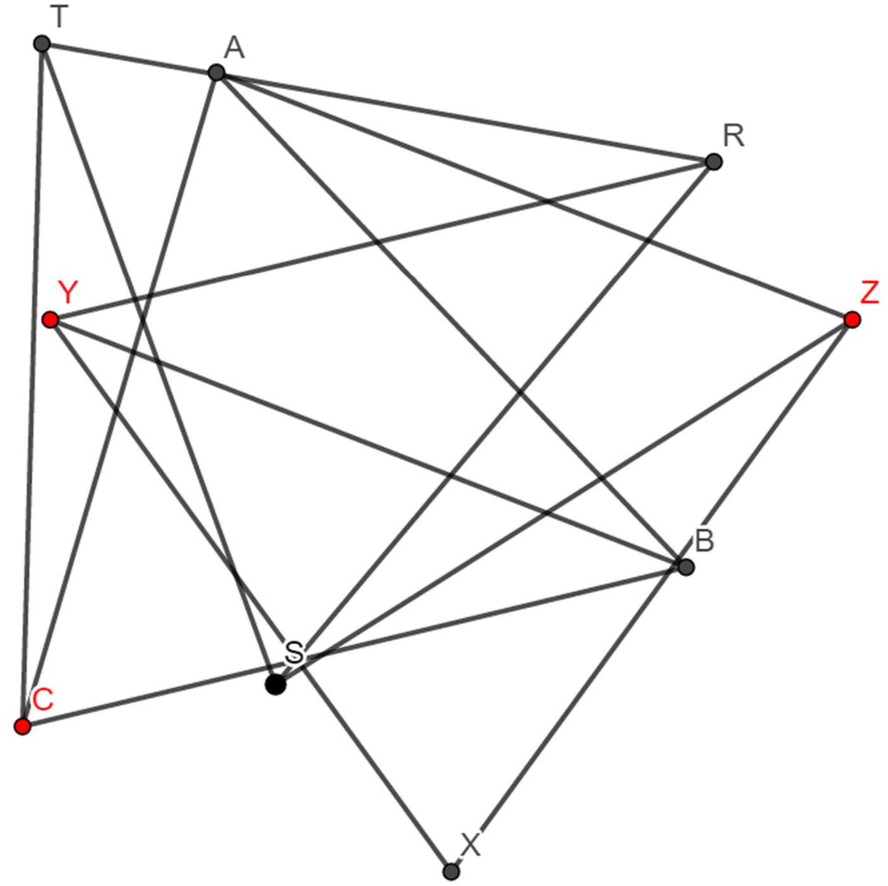
Finally~~, for each C_i,~~ we add a~~two~~ “Golomb triangle~~s~~”: RST~~one~~ for Y, Z ~~C_i, Y~~ and C~~Z and the other for C_i, A and B~~. Here, a Golomb triangle for points L, M and N is a six-vertex, six-edge UDG consisting of a unit triangle RST with R,S,T connected to L,M,N respectively (see Figure xxx); the name is inspired by the 10-vertex, 4-chromatic graph known as the Golomb graph in which L,M,N are the vertices of an equilateral triangle of edge Sqrt[3]. Depending on the relative locations of L,M,N, there are between zero and 12 possible locations for R,S,T. For present purposes, since the distances YZ~~LM~~, YC~~LN~~ and ZC~~MN~~ asymptotically approach 1 as n rises~~are always all close to 1~~, it is easy to see that R,S,T can be chosen to lie close to Z,C,Y~~M,N,L~~

respectively (see figure). ~~We will not take the trouble to name the vertices of our Golomb triangles, because they serve only one purpose for us: in a 3-coloring of the graph LMNRST, it is not possible for L, M and N to be all the same color.~~

Now we can start to assign colors to our vertices. In any proper 3-coloring, s~~S~~ince n is not a multiple of 3, the ncycle with which we began must have at least one vertex whose neighbors (in the cycle, not the polygon) are the same color. We designate such a vertex to be the vertex X in the construction just described. Without loss of generality we assign X color 1 and both Y and Z color 2; this forces C~~both A and B~~ to be color 2~~3~~.

Consequently, none of R, S and T can be color 2.~~Next, notice that all C_i must be color 1. They cannot be color 2, because there is a Golomb triangle connecting them to Y and Z, and similarly they cannot be color 3, because there is a Golomb triangle connecting them to A and B.~~

~~Now we consider the vertices P_i, in increasing order. P_1 is connected to Z (which is color 2) and C_1 (which is color 1), so it must be color 3. Then P_2 is connected to P_1 (which is color 3) and C_2 (which is color 1), so it must be color 2. Continuing, we see that all P_i with I odd receive color 3. But j+1 is odd, so Y receives color 3, which contradicts our original assignment of color 2 to Y.~~ Thus, our graph is not 3-colorable.


T
A
R
Y
Z
B
S
C
X