FROM GRASSMANN TO CLIFFORD

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ABSTRACT

The aims of this note are to convince the readers in as elementary as possible way that:
1. Clifford algebra is the *particular* case of the Grassmann algebra which is the most fundamental from the point of view of the physics, geometry and analysis. Grassmann algebra with the (pseudo-) Riemannian structure (including degenerate case), or with the (pre-) symplectic structure, or with the Hermitian (Hilbert) structure, contain the corresponding Riemannian or symplectic or Hermitian Clifford algebra.
2. Grassmann algebra is the tautology of the formalism of the (multi-) fermion and antifermion creation and annihilation operators. This relation does not need at all to fix any kind of the Riemannian (Euclidean) or Hilbertian, Hermitian, etc, structures.
3. Clifford algebra is the tautology of the formalism of the quasi-particles in nuclear physics and therefore can be viewed as the Bogoliubov transformation of the Grassmann algebra.
4. The last two statements open wide possibilities of the applications of the Grassmann and Hermitian Clifford algebras particularly to nucleons and quarks in the nuclear shell theory.
5. The Clifford product can always be reexpressed in terms of the fermion creation and annihilation operators.
6. It is interesting that symplectic Clifford algebras has not yet been explored in the classical Lagrangian and Hamiltonian mechanics.

The pair of the mutually dual linear spaces and modules. Whenever we consider or deal with some linear space L, over say field $\mathbb{R}$ or $\mathbb{C}$, with elements having this or another physical interpretation then we have on hand automatically, independent of our wishes, another linear space $L^*=\mathrm{Hom}(L,\mathbb{R})$, dual to L, of all linear mappings

$$L^* \ni \alpha: L \ni v \to \alpha v \in \mathbb{R}. \tag{1}$$

Whenever L has a physical interpretation, then L^* has also a physical interpretation. Roughly speaking L^* is the set of the measurement apparatuses and αv in the above (1) is a result of an orientation-dependent measurement of v by means of the instrument α (unfortunately we do not

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take into account that any physical measurement has finite accuracy). In the finite dimensional case (which for simplicity we will consider here) $\dim L = \dim L^* < \infty$. Also both spaces L and L^* are mutually dual to each other, $L^{**} = L$, and within the algebra are not distinguished, and can *never* can be identified! As everybody knows, one member of this pair $\{L, L^*\}$, say L, naturally gives rise to the family of contravariant tensors, and another is generating the family of covariant tensors, the names of course having only historical meaning. The modelling of the physical quantities by the elements of L or L^* is not simply a convention but comes from experimental observations as first was realized by James Clerk Maxwell in his Treatise in 1864. Introduction of the Euclidean structure or any other bilinear form $\phi: L \times L \to \mathbb{R}$, greatly obscures discrimination between L and L^* forcing identification of these spaces. To say that we do not need to consider "any" dual space is meaningless because, independently of our wishes, dealing with L we have to do at least with the pair $\{L, L^*\}$. In fact all tensor products are involved. However, to say that we do not need to fix any particular Euclidean structure is meaningful. If we consider the particular linear space L, then of course we have on hand naturally the space of all bilinear mappings $L \times L \to \mathbb{R}$, i.e. the space of the second degree covariant tensors. But of course this is not the same as fixing the particular member of this space. For what follows it is important to understand the geometrical representation of the covector $\alpha \in L^*$ as the codimension one hyperplane $\mathrm{Ker}\alpha \equiv \{v \in L, \alpha v = 0\} \subset L$. In fact the covector α is completely and uniquely determined by $\mathrm{Ker}\alpha$ and any vector $v \in L$ such that $\alpha v = 1$. The wave fronts for instance are described by covectors (forms).

The above without change is also true for the pair of the mutually dual modules $\{V^1, \Lambda^1\}$ generated by the commutative algebra F. The commutative algebras, for instance the algebra of the real-valued smooth functions on some smooth manifold M, serve as the model of the set of the classical (exact) measurements. In what follows, however, we do not need to pick up any of the particular functional realizations of the given algebra F as the algebra of the functions on some manifold. In fact we do not need the notion of a manifold, because analysis on the manifolds is and ought to be fully contained in a set V^1 of derivations on commutative algebras. This set of derivations V^1 is just the F-module of the vector fields. Dual to V^1, is the F-module Λ^1 of the differential forms, so $\alpha \in \Lambda^1$ is an F-linear mapping

$$\alpha : V^1 \ni x \to \alpha x \in F. \tag{2}$$

Also here within the algebra the modules V^1 and Λ^1 are not distinguished; however, analysis is clearly discriminating them because only vector fields are generating the flows (evolutions) on algebra F and only differential forms could be integrated, and so on.

Therefore there is no advantage or simplification if one does not need to talk about the pair of the mutually dual linear spaces or modules and instead is using only one linear space of vectors or one module of fields. In most cases the price is the obscure identification of L^* with L.

Grassmann multiplications. The geometrical and physical meaning of the Grassmann multiplications (exterior and interior) can *not* be understood if we don't wish to consider the dual pair $\{L, L^*\}$ or $\{\Lambda^1, V^1\}$. If $\alpha^i \in \Lambda^1$ and $x_i \in V^1$ for $i \in \{1,2,\ldots\}$, then both the exterior Grassmann multiplication of the forms and the exterior Grassmann multiplication of the vectors (the wedge products $\wedge$) are *defined* through the determinants, having the clear geometrical meaning as oriented volumes, viz

$$(\alpha^1 \wedge \ldots \wedge \alpha^k)(x_1 \wedge \ldots \wedge x_k) \equiv \det\{\alpha^i x_j\} \quad . \tag{3}$$

Evidently (3) generalizes the $\mathbb{R}$- or F-linear mappings (1-2). Here $\alpha^1 \wedge \ldots \wedge \alpha^k \in \Lambda^k$ is called a decomposable (differential) form of degree k, etc. The decomposable multiforms and decomposable multivectors are generating linearly *two* copies of the Grassmann algebras V and Λ of the multivector fields and of the differential multiforms. We put $V^0 = \Lambda^0 \equiv F$. All algebraic properties of the Grassmann products $\wedge$ are equivalent to and follow from the theory of determinants [1]. In particular the Grassmann exterior products are associative (theorem!).

It is important to realize that the formula (3) has nothing to do with the scalar product, because it is independent of any Euclidean structure. Also determinant is the *value* of the form and not the form itself. With this respect cf. with [5] (formula (3.17)) and with ([6], Ch.1-4) where the definitions are essentially different.

The Grassmann exterior product

$$\wedge : V \times V \ni x, y \to x \wedge y \in V \tag{4}$$

(and similiar for the multiforms) where $\deg(x \wedge y) = \deg x + \deg y$, is much better represented by the equivalent tensor operator e of the left multiplication (also known as the left adjoint representation)

$$V \ni x \overset{e}{\to} e_x \in \operatorname{End} V ,$$

where $\quad e_x y \equiv x \wedge y \quad .$ (5)

$$\text{Then:}\quad e_x \circ e_y = (-)^{(\deg x)(\deg y)}\, e_y \circ e_x \tag{6}$$

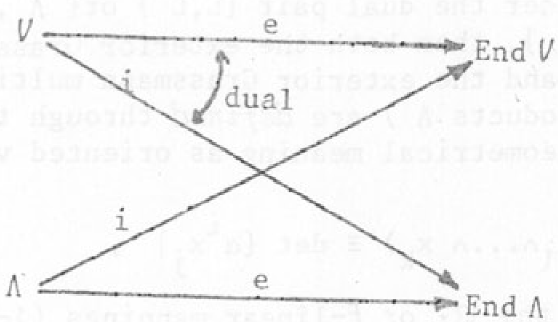
$$(\Rightarrow (e_x)^2 = 0 \quad \text{for } \deg x = \text{odd}).$$

The associativity of the exterior Grassmann products $\wedge$, means that

$$e_x \circ e_y = e_{e_x y} \tag{7}$$

and has nothing to do with the obvious associativity in End V: $e_x \circ (e_y \circ e_z) = (e_x \circ e_y) \circ e_z$.

The pair of the Grassmann algebras $\{V, \Lambda\}$ is interrelated through the interior Grassmann products "i", dual, by definition, to the exterior Grassmann products $e, i \equiv$ pull back of e, all of which are F-linear mappings



(8)

Here

$$(i_x\omega)y \equiv \omega(e_x y) \in F$$

for $\deg\omega = \deg x + \deg y$,

and $\beta(i_\alpha x) \equiv (e_\alpha\beta)x \in F$

for $\deg x = \deg\alpha + \deg\beta$. (9)

$(\alpha,\beta,\omega \in \Lambda$ and $x,y \in V)$.

Also $\deg(i_x\omega) = \deg\omega - \deg x$

and $i_x\omega = 0$ for $\deg\omega < \deg x$, etc. (10)

(One can consider instead of (9) also the right as well as the left Grassmann exterior products and their duals). From (9) it follows that

$$i_x \circ i_y = (-)^{(\deg x)(\deg y)}\, i_y \circ i_x \tag{11}$$

$$(\Rightarrow (i_x)^2 = 0 \quad \text{for } \deg x = \text{odd}),$$

and $i_x \circ i_y = i_{e_y x}$. (12)

The above discussion shows clearly that the names exterior and interior should in fact be interchanged. For example it is meaningless to ask about the associativity of the interior product.

In the mathematical literature the interior Grassmann product usually is introduced as the antiderivation of the (abstract) Grassmann algebra with respect to the involution $\alpha \to (-)^{\deg\alpha}\alpha$ (see e.g. [4] Ch.V § 4). However, here this property of the interior products is simply a consequence of the definitions (3) and (9). In fact one can easily show that for instance for $\deg x = 1$

$$i_x \circ e_\alpha = e_{i_x \alpha} + (-)^{\deg\alpha} e_\alpha \circ i_x . \tag{13}$$

Here i_x and $e_\alpha \in \text{End}\,\Lambda$. We prefer, however, to present relations similar to (13) (including arbitrary degx) in another form more suitable for the proper physical interpretation. Namely, let's define the *superbracket*

$$\{A,B\} \equiv A \circ B - (-)^{(\deg A)(\deg B)} B \circ A \tag{14}$$

for $A, B \in \text{End}\,\Lambda$ or End V. Then for instance in End Λ we have the following relations (cf with (6) and (11))

$$\{e_\alpha, e_\beta\} = 0$$

$$\{i_x, i_y\} = 0$$

$$\{i_x, e_\alpha\} = e_{i_x \alpha} + (\ldots) \tag{15}$$

In particular, if the degrees of all multiforms and multivectors in (15) are equal to one, then the superbrackets are simply the anticommutators; and because

$$e_{i_x \alpha} = \alpha x ,$$

then

$$\{i_x, e_\alpha\} = \alpha x \in F \tag{16}$$

(here degx = degα = 1). In this way we recover the familiar anticommutation relations for the fermion creation and anihilation operators (15-16). We see also that what is called in mathematics an antiderivation of the Grassmann algebra is the same as an fermion anihilation operation in particle physics. However, in order to have the proper physical interpretation of the Grassmann products e and i in (8) we must identify the pair of the mutually dual Grassmann algebras $\{V, \Lambda\}$ with the Fock spaces of the fermions and anti-fermions. If e is the multi-fermion creation operator then $i = e^*$ is the multi-*anti*-fermion anihilation operator and vice-versa.

The terms on the right hand side of equation (15) denoted by dots indicate the violation of the ideal fermions and bosons (anti-) commutation relations if these particles are built up from fermions. The boson-fermion duality and construction of bosons from fermions can be achieved only in ∞ - dimensional spaces. For example if degx=degy=1 and degα=2 then the *commutator* of the creation and anihilation operators of the pairs of (anti-) fermions has the explicit form

$$[i_{x\wedge y}, e_\alpha] = \alpha(x \wedge y) + e_{i_y \alpha} \circ i_x - e_{i_x \alpha} \circ i_y , \tag{17}$$

which shows that pair of fermions is not an ideal boson.

In what follows we will understood the Grassmann algebra as the pair $\{V,\Lambda\}$ with the full set of the exterior and interior (right and left) Grassmann products (8).

Quasi-particles, super-operators and Dirac relation. Because e_α and $i_x \in \mathrm{End}\,\Lambda$ (see (8)) for all α in Λ and all x in V, then we have within this pure Grassmann algebra also the quasi-(anti-) particle operators (nuclear physicist's terminology)

$$e_\alpha + i_x \in \mathrm{End}\,\Lambda,$$

and their pull-back's (duals),

$$(e_\alpha + i_x)^* = e_x + i_\alpha \in \mathrm{End}\,V. \tag{18}$$

Therefore having the Grassmann algebra $\{V,\Lambda\}$, (8), we can consider the new algebra $V\times\Lambda$ with the multiplication taken as the quasi-particle operator and denoted by γ,

$$V \overset{\oplus}{\times} \Lambda \ni \{x,\alpha\} \xrightarrow{\ \gamma\ } \gamma_{x,\alpha} \in \mathrm{End}(V\times\Lambda). \tag{19}$$

Here

$$\gamma_{x,\alpha}\{y,\beta\} \equiv \{(e_x + i_\alpha)y, (e_\alpha + i_x)\beta\}. \tag{20}$$

This new algebra $\{V\times\Lambda, \gamma\}$ is the clear generalization of the Kähler-Atiyah algebra [7],[2] and [3]. Extremly important is here that we do *not* fix yet any kind of the correlation between V and Λ. Choosing particular Riemannian structure g (see below) we get from (20) exactly the original Kähler-Atiyah multiplication. However, our multiplication (20) does *not* depend on either Riemannian or symplectic structures, etc.

Consider only *odd* degree elements in Grassmann algebra (8). The Grassmann multiplications e and i of *odd* degrees are called *super*-operators or *super*-charges; here $\deg e_\alpha = \deg\alpha$ and $\deg i_x = -\deg x$. Then the anticommutator of the Kähler-Atiyah super-operators (20) (restricted say to V) obey

$$\{\gamma_{x,\alpha}, \gamma_{y,\beta}\} = \{i_\alpha, e_y\} + \{i_\beta, e_x\}. \tag{21}$$

This looks like the celebrated relations in say supersymmetric quantum mechanics [8] and super-quasi-particles operators γ are analogous to the super-charges, except that we do not used any kind of the correlation between Λ and V, and relation (21) is independent of either Hilbert or Riemannian structures.

Also it should be clear that (21) generalize the Dirac relations for the familiar Dirac matrices. However, the γ-"matrix" now represents a *pair* of a, say, vector x and a covector α, which are independent. For example γ now represents (in a set $\mathrm{End}(V\times\Lambda)$) a space-time splitting on space $\{x\}$ and time $\{\alpha\}$ *without* pseudo-Riemannian (Minkowskian) structure.

In particular, from (21) it follows that

$$(\gamma_{x,\alpha})^2 = \{i_\alpha, e_x\}, \tag{22}$$

which means that the super-operators are the square-roots of the anti-commutators. Combining (22) with eq.(16) for degx=degα=1, we have

$$(\gamma_{x,\alpha})^2 = \alpha x \in \mathbb{R} \quad \text{or } F \quad .$$

The point is that the multiplication (20) has the unpleasant feature of non-associativity like the original Kähler-Atiyah Riemannian multiplication. Namely for the associator we get (cf. with eq.(8)),

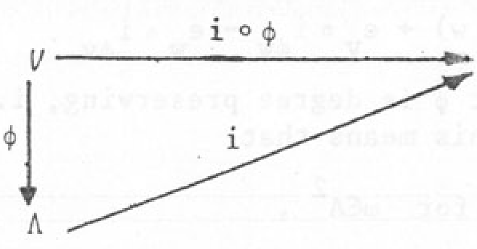
$$\gamma_{x,\alpha} \circ \gamma_{y,\beta} - \gamma_{\gamma_{x,\alpha}\{y,\beta\}} = i_\alpha \circ i_\beta - i_{e_\alpha \beta}$$
$$+ i_\alpha \circ e_y - e_{i_\alpha y}$$
$$+ e_x \circ i_\beta - i_{i_x \beta} .$$

This hold true for arbitrary degrees.

<u>Clifford algebras.</u> Suppose ϕ is any (semi-) F-linear mapping

$$\phi : V \to \Lambda , \qquad ,$$

induced for example from the Poincaré isomorphism ([4], Ch.VI § 2) or from the (pseudo-) Riemannian, (pre-) symplectic or Hermitian structures (not necessarily invertible: similarly one can consider the mapping $\Lambda \to V$). For instance $\phi = g + \omega$, where g is a (pseudo-) Riemannian structure and ω is a bi-form. Both cases of the Riemannian *and* symplectic Clifford algebras can be considered jointly.



Then we can introduce the *Clifford multiplication* in the set V as F-linear mapping,

$$V \ni v \overset{\gamma}{\to} \gamma_v \in \text{End } V$$

which we *define* as follows

$$1.\quad \gamma_v = e_v \qquad \text{for } \deg v = 0,$$

$$2.\quad \gamma_v = (e + i\circ\phi)_v \qquad \text{for } \deg v = 1,$$

$$3.\quad \gamma_{\gamma_v w} = \gamma_v \circ \gamma_w \qquad \text{(associativity).} \tag{23}$$

The above conditions determine γ uniquely and the set of the multivector fields V with the multiplication γ (23) is the Clifford algebra. Both Riemannian and symplectic Clifford algebras defined in this way have the *same* dimension (=dim V).

We like to show that the Clifford product (23) can be reexpressed in terms of the fermion creation and annihilation operators. This one can show through the induction on the decomposable elements. If degv=1 then

$$\gamma_v w = v \wedge w + i_{\phi v} w ,$$

and the associativity condition gives

$$\gamma_{\gamma_v w} = \gamma_{v\wedge w} + \gamma_{i_{\phi v} w} = \gamma_v \circ \gamma_w .$$

This shows that

$$\gamma_{v\wedge w} = \gamma_v \circ \gamma_w - \gamma_{i_{\phi v} w} \qquad (\deg v = 1). \tag{24}$$

The above means that the associativity of the Clifford product imply that each homogeneous Clifford operator can be expressed by means of the Clifford operators of smaller degrees. For example for degw=1 in (24) we get

$$\gamma_{v\wedge w} = \gamma_v \circ \gamma_w - \phi(v,w) . \tag{25}$$

This can be evaluated further:

$$\gamma_{v\wedge w} = (e - i\circ\phi)(v\wedge w) + e_v \circ i_{\phi w} - e_w \circ i_{\phi v} .$$

In (25) we tacitly assumed that ϕ is degree preserving, i.e. that $\deg\phi w = \deg w$ for all w in V. This means that

$$\phi = g + \omega \qquad , \text{ for } \omega\in\Lambda^2 . \tag{26}$$

Eq.(25) one can present through the commutator,

$$\gamma_{v\wedge w} = \frac{1}{2}\left[\gamma_v, \gamma_w\right] - \omega(v\wedge w) ,$$

because anticommutator is independent of the (pre-) symplectic part of ϕ (26),

$$\{\gamma_v, \gamma_w\} = 2g(v,w) \quad ,\text{for } \deg v = \deg w = 1.$$

These formulae indicate clearly once more why the Clifford multiplication γ should *not* be considered as the primitive concept (along the lines developed in [5] and [6]). Most natural is the definition of the Clifford multiplication γ (23) in terms of the primitive Grassmann multiplications e and e^* with some structure ϕ. In fact Clifford algebra is a pair: Grassmann algebra *and* some ϕ. Putting $\phi=0$ is *not* the same as not choosing any structure ϕ at all. We consider the definition of the Grassmann multiplication e in terms of the Clifford multiplication as completely inappropriate.

Let k,l,m,p,q,r,s be vectors in V^1. Then for the degree preserving mapping ϕ (26) we have

$$\gamma_{k\wedge l\wedge m} = \gamma_k \circ \gamma_l \circ \gamma_m$$

$$+ \phi(k,m)\gamma_l - \phi(l,m)\gamma_k - \phi(k,l)\gamma_m .$$

$$\gamma_{p\wedge q\wedge r\wedge s} = \gamma_p \circ\gamma_q \circ\gamma_r \circ\gamma_s$$

$$+ \phi(q,s)\gamma_p\circ \gamma_r - \phi(r,s)\gamma_p \circ\gamma_q - \phi(q,r)\gamma_p \circ\gamma_s$$

$$- \phi(p,q)\gamma_r \circ \gamma_s + \phi(p,r)\gamma_q \circ \gamma_s - \phi(p,s)\gamma_q \circ \gamma_r$$

$$+ \phi(p,q)\phi(r,s)-\phi(p,r)\phi(q,s)+\phi(p,s)\phi(q,r). \qquad (27)$$

The above formulae are sufficient for the effective applications of the Clifford calculus for the four dimensional spacetime. Eq.(26) clearly could be interpreted in framework of the Einstein nonsymmetric unified theory of the gravitational $\{g\}$ and electromagnetic $\{\omega\}$ fields.

One can ask: why we are postulating the associativity in (23)? The algebra can be defined by means of any other condition on associator. The most important notion related to associative Clifford algebras is a spinor space as minimal ideal in algebra. One can say that the spinors imply an associativity. From (23) it follows that

$$\gamma_p \circ \gamma_p = \gamma_p \Leftrightarrow \gamma_p\, p = p \in V \; . \qquad (28)$$

Suppose that p is a primitive idempotent, i.e. $\gamma_p p=p$ and p is generating the left minimal ideal $I_p \subset V$ in Clifford algebra. Then the Dirac operator Γ_p in the spinor space I_p is explicitly p-dependent,

$$V \ni v \xrightarrow{\Gamma_p} \delta_p \circ \gamma_v = \gamma_v \circ \delta_p \in \mathrm{End}\, I_p$$

Here $\delta_a b \equiv \gamma_b a$, i.e. δ defines the multiplication opposite to γ[4] .

We have

$$\gamma_a \circ \gamma_b = \gamma_{\gamma_a b} \Leftrightarrow \delta_a \circ \delta_b = \delta_{\delta_a b},$$

Therefore (28) is equivalent to

$$\delta_p \circ \delta_p = \delta_p \Leftrightarrow \delta_p p = p. \tag{29}$$

The opposite product δ can also be expressed in terms of the fermion creation and anihilation operators e and e^*. Choosing the particular basis in I_p we get the particular matrix representation of the Dirac operator Γ^p. However, this matrix representation is completely irrelevant. What *is* essential is the p-dependence of the Dirac operator dependence which is mostly ignored or not noticed. One should stress that the idempotent p is a multifield on a manifold and not every manifold allows the global existence of such a multifield: this is the reason why spinors do not exist globally on arbitrary manifolds.

Suppose that the idempotent p is of the simple form

$$2p = \lambda + v + a \wedge b, \tag{30}$$

where degλ=0 and v,a and b are vectors in V^1.
Then eqs.(28-29) imply that (using (26) and for p≠1)

$$v \wedge a \wedge b = 0,$$

$$\lambda = 1 + \omega(a \wedge b),$$

$$v\omega(a \wedge b) + b\omega(v \wedge a) - a\omega(v \wedge b) = 0,$$

$$g(v,v) + [g(a,b)]^2 - g(a,a)g(b,b) = 1. \tag{31}$$

One thing this shows is how important are a Riemannian structure and its signature (for g=0 there are no nontrivial idempotents, at least of the form (30)). From (30-31) we have two examples.

$$A = \frac{1}{2}(1 + v) \quad \text{with } g(v,v) = +1,$$

and $$B = \frac{1}{2}(1 + \omega(a \wedge b) + a \wedge b) \quad \text{with}$$

$$[g(a,b)]^2 - g(a,a)g(b,b) = 1.$$

These idempotents A and B are not necessary primitive ones: we do not yet fix the dimension. If they commute,

$$\delta_A \circ \delta_B = \delta_B \circ \delta_A, \tag{32}$$

then $\delta_A B = \delta_B A$ is again an idempotent. Neglecting ω in (31) we get from (32) that $g(v,a)=g(v,b)=0$ (ω in (31) could be interpreted as the distortion of the spinor space through the electromagnetic field). For the case of the four dimensional space-time the idempotent $\delta_A B$ is primitive one and is determined, as we see, by the complete rèpere mobile (vierbein), because in this case $v \wedge a \wedge b \neq 0$.

If $\phi: \Lambda \to V$ we can then build the Clifford algebra of the differential multiforms. From superbrackets

$$\{d, e_\alpha\} = e_{d\alpha}$$

$$\{d, i_x\} = L_x$$

$$\{L_x, i_y\} = i_{L_x y}\,, \qquad \text{etc}$$

(for all α in Λ and all x,y in V) it follows that

$$\{d, \gamma_\alpha\} = e_{d\alpha} + L_{\phi\alpha} \qquad \text{for } \deg\alpha = 1.$$

Therefore if $\beta \in \Lambda$ is idempotent then the exterior derivative operator d does *not* commute with the spinor projection: $d \circ \delta_\beta \neq \delta_\beta \circ d$.

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REFERENCES

[1] V. Arnold *Les Méthodes Mathématiques de la Mécanique Classique* Mir: Moscou (Russian edition: 1974)§32-33,1974
[2] M.F. Atiyah, R. Bott and A.A. Shapiro *Clifford Modules, Topology* 3 (Supp.1)3-38,1964
[3] I.M. Benn and R.W. Tucker "The Dirac equation in exterior form" *Commun.Math.Phys.* 98, 53-63,1985
[4] W.H. Greub *Multilinear Algebra* Springer-Verlag: New York, 1967
[5] D. Hestenes *Space-Time Algebra* Gordon and Breach: New York 1966
[6] D. Hestenes and G. Sobczyk *Clifford Algebras to Geometric Calculus* D. Reidel Publishing Company: Dordrecht, 1984.
[7] E. Kähler *Rendiconti di Matematica* 3-4, 21,425,1962
[8] E. Witten "Supersymmetry and Morse Theory" *J. Differential Geometry* 17, 661-692,1982

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