Fullerenes, Polyhedra, and Chinese Guardian Lions

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EUGENE A. KATZ AND BIH-YAW JIN

Does your hometown have any mathematical tourist attractions such as statues, plaques, graves, the café where the famous conjecture was made, the desk where the famous initials are scratched, birthplaces, houses, or memorials? Have you encountered a mathematical sight on your travels? If so, we invite you to submit an essay to this column. Be sure to include a picture, a description of its mathematical significance, and either a map or directions so that others may follow in your tracks.

➢ Submissions should be uploaded to http://tmin.edmgr.com or sent directly to **Dirk Huylebrouck,** huylebrouck@gmail.com

We, the coauthors of this paper, met for the first time in 2014 at the 2nd International Conference "Science, Technology, and Art Relations" in Tel Aviv. In his lecture "Fullerene-like Architecture in Nano-, Micro-, and Macro-worlds" Eugene Katz showed his photographs of two statues of guardian lions at the Forbidden City in Beijing and mentioned that the lions' paws rest on balls, which, combinatorially, are fullerene polyhedra and their duals. Bih-Yaw Jin challenged him on several points. This short discussion spurred our collaboration on this article, and it became a report on a mathematical tourist detective investigation.

Statues of Chinese guardian lions, known as Shishi for stone lions or Tongshi for bronze lions, stand in front of Chinese Imperial palaces, Imperial tombs, government offices, temples, and the homes of government officials and the wealthy; the tradition dates back to the Han Dynasty (206 BC–AD 220). The lions are believed to have powerful mystical protective powers. They are always presented in pairs, a manifestation of *yin* and *yang*, the female representing yin and the male yang. The female has a cub under the left paw, representing the cycle of life. The male lion is essentially identical, but has its right front paw on an embroidered ball. Sometimes the ball is carved with a spherical network that is similar to that of a fullerene polyhedra or a fullerene dual. (A fullerene polyhedron is one with only pentagonal and hexagonal faces. It is easy to show that no matter how many hexagonal faces such a polyhedron has, the number of pentagonal faces must be twelve. For more about fullerene polyhedra and their duals, see the Appendix to this paper.)

One of us (E.K.) first learned about the fullerene-like decorated sphere under the paw of the guardian lion in front of the Gate of Heavenly Purity (the Qianqing Gate) at the Forbidden City in Beijing from Istvan Hargittai's article [1]. Hargittai seems to have been the first scientist to pay attention to this pattern. On his first visit to Beijing, E.K. wanted to find and photograph it. He did (Fig. 1a–c), and he also found, surprisingly, the paw of a male lion in front of the Hall of Spiritual Cultivation (Yangxing Dian) on a ball with the fullerene dual structure (Fig. 1b–d). E.K. rashly decided that these statues were erected in the same period as the Forbidden City was built (Early Ming Dynasty, ~1420).

B.-Y.J. questioned this dating. The buildings in the Forbidden City are made of wood, and many of them have been burned down and rebuilt. The palace behind the bronze lion has been rebuilt several times. Even though the bronze lion could not be burned down so easily, the statues should be dated according to historical records.

B.-Y.J. also queried the fullerene structure, having earlier compared it to a picture he found on the Internet

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Figure 1. Male gilded guardian lions at the Forbidden City (a, b) and the balls under their right front paws (c, d). The balls are carved with a pattern that resembles a fullerene (a, c) and a fullerene dual (b, d). Beijing, China. Photograph by E. A. Katz.

(Fig. 2). If the structure had icosahedral symmetry, the centers of the two pentagons (marked in red) would be the vertices of an equilateral triangle whose third vertex would be the center of a third pentagon at the location of the white dot. However, there is a hexagon at that position.

Analysis of the "dual fullerene" (Fig. 1b–d) showed three pentagonal arrangements of triangular faces (or three vertices of degree five) on its front side; see the red pentagons in Figure 3. (Goldberg vectors (2,1) and (4,0) could be identified; see the Appendix).

We agreed that these structures do not have icosahedral symmetry. But do they have a structure of any other fullerene (or fullerene dual)? In other words, do they have 12 pentagonal faces (or 12 pentagonal connections)? To answer this question one should examine the entire surface of the ball.

B.-Y.J. studied the Chinese literature [2] and learned that near or on each sculpture in the Forbidden City there should be information about its dating. He asked his former postdocs, Yao Shen and Qing Ai, now working in Beijing, to visit the Forbidden City to check this information and to photograph the balls from various perspectives. Later, he also visited the Forbidden City and carefully examined the three Bronze lion statues with claws stepping on "fullerene-like" balls in front of Qianqing Gate, Yangxing Dian, and Ningshou Gate (Gate of Tranquil Longevity).

The Forbidden City was built in the early fifteenth century ($\sim$1420–1644), but all these lion balls decorated with "fullerene-like" structures were constructed much later, during the Emperor Qianlong's reign in the Qing dynasty (1740–1800), an era of prosperity in China. Looking at the ball in front of the Qianqing Gate from various sides, one finds only four pentagonal faces. Although it is not possible to look at the ball from its backside, it is unlikely that this small part contains the eight missing pentagons. Closer to the bottom of the ball, the tiling pattern becomes quite irregular (Fig. 4a). Furthermore, one can identify two edge-sharing triangles and also a distorted hexagon sharing a

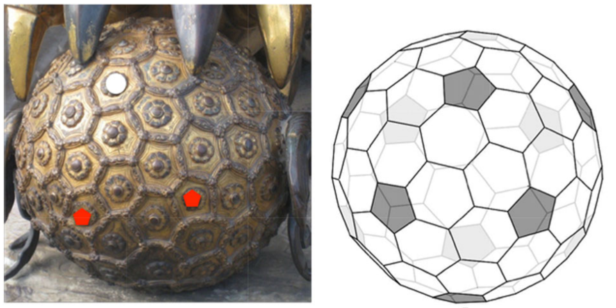

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Figure 2. (a) The decorated ball shown in Figures 1a–c; (b) fullerene C_{140} with the Goldberg vector (2,1). (For definition of Goldberg vector, see the Appendix to this article.)

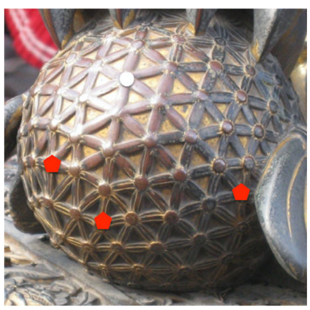

Figure 3. Analysis of the "dual fullerene" shown Figure 1d. Red pentagons mark the locations of the pentagonal connections of the triangular faces (or vertices of degree five). The white dot marks the six-degree vertex where the five-degree should occur.

single edge with two other polygons. Thus, we can safely say that this is not a fullerene structure.

The ball with triangular tiling (Fig. 1b–d) located at Yangxin Dian certainly does not have the structure of a fullerene dual, since Yao Shen and Qing Ai found some vertices of degree seven (marked by blue heptagons in Fig. 4b).

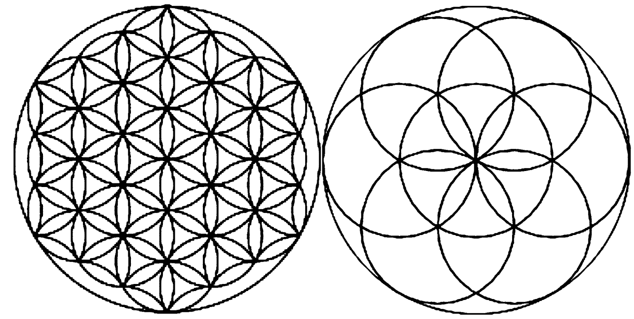

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Figure 5. Flower of Life (a) and Seed of Life (b).

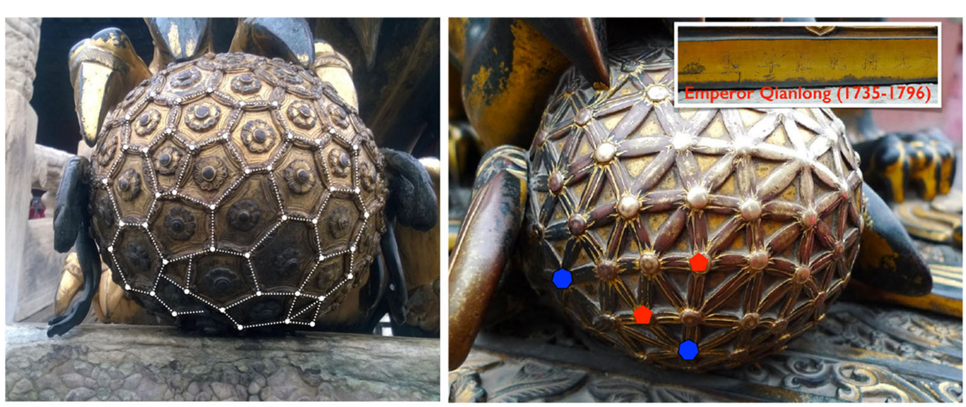
The name "Flower of Life" is given to a plane tiling composed of multiple evenly spaced, overlapping circles (Fig 5a). The center of each circle is on the circumference of six surrounding circles of the same diameter. This flat figure forms a flowerlike pattern with so-called *Seed of Life* as basic component of such tiling. The latter consists of seven circles being placed with sixfold symmetry (Fig. 5b). It is an easy consequence of Euler's formula

$$F - V + E = 2$$

however that this pattern cannot be inscribed on a sphere.

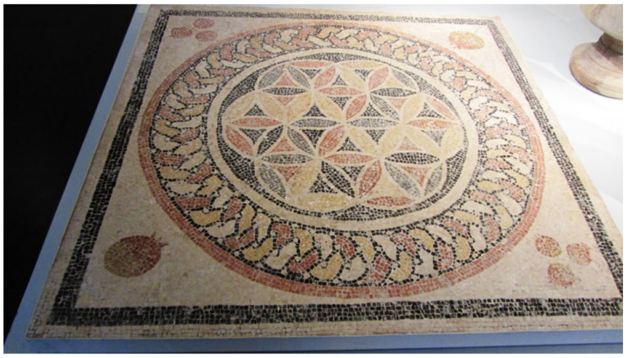
The Flower of Life pattern has symbolic meaning in many cultures and religions worldwide. For example in the Judeo-Christian tradition, the Seed of Life is supposed to symbolize the seven days of creation. Examples of these patterns can be found in the ancient Egyptian, Phoenician, Assyrian, Indian, Asian, Middle Eastern temples, and in medieval art. Figure 6 depicts ancient artifacts with the Flower of Life and the Seed of Life found in Israel.

In the Forbidden City, at the Gate of Supreme Harmony (the Taihe Gate), Yao Shen and Qing Ai found another sculpture of a lion with a ball tiled in a "Flower of Life-like" pattern (Fig. 7). This was also constructed in the Qianlong period. One cannot find any pentagonal connection on the ball surface; evidently the anonymous sculptor did not perform a proper geometrical analysis.



a b

Figure 4. (a) Molecular "graph" on the "Fullerene-like" structure. (b) Blue heptagons indicate locations of the heptagonal connection of the triangular faces on the surface of the ball with triangular tiling. Photograph by Yao Shen and Qing Ai.



a

b

Figure 6. Ancient artifacts with the Flower of Life (a) and the Seed of Life (b) in Israel: (a) Stone mosaic floor from the Herod's Palace at Herodium, 1st century BCE, Israel Museum, Jerusalem; (b) Decorated altar with basin on top from the Palace of Agrippa, Banias, 1st century CE. Photographs by E. A. Katz.

Finally, we note that a guardian lion statue with the "fullerene-dual-like" decorated sphere under its paw has already been mentioned in scientific literature [3]. This one is located in front of the East Gate, the main entrance of the Summer Palace (Yiheyuan garden) in Beijing. One can find three vertices with five connections and a few Goldberg vectors (see the Appendix) connecting them ((2,1) and (3,2)). Another pair of bronze lion statues in the Summer Palace is located in front of Paiyun Gate. The decoration on the ball under the male lion statue's paw also has a pattern of "Flower of Life" with no pentagons. According to the historical records, these two pairs of bronze lion statues were both built in Qianlong's reign in China (1736–1795). However, the pair in front of Paiyun Gate was originally located in the main entrance of the Old Summer Palace (Yuanmingyuan Garden) [4].

Figure 7. Ball tiled with a structure resembling the "Flower of Life" under the paw of the Lion located at the Taihe Gate, Forbidden City, Beijing. Photograph by Yao Shen and Qing Ai.

The truncated icosahedron is one of the 13 Archimedean bodies referred to by Pappus of Alexandria (AD 290–350), as well as a member of the fullerene family. The oldest published image of the truncated icosahedron can be found in the book "De Divina Proportione" (printed in 1509) [5], written by Luca Pacioli (1445–1514) and illustrated by Leonardo Da Vinci. However, Pacioli is often blamed for plagiarism of unpublished manuscripts of his mathematical teacher, the great artist Piero della Francesca (~1415–1492) [6], who nowadays is given credit for the rediscovery in the West of the truncated icosahedron and some other Archimedean bodies [7]. His manuscript *Libellus de Quinque Corporibus Regularibus (Short Book on the Five Regular Solids*), written around 1480, probably has the oldest convincing image of the truncated polyhedron [8].

Anyway, one can conclude that, the "fullerene-like" balls shown in Figure 1 are much younger than Piero's drawing and they are not mathematically correct. It seems that their creator never built the entire sphere. However, a number of questions for historians of science and art remain open and still require additional investigation. Two of the most important are: (1) Were these beautiful "fullerene-like" artefacts subjected to mathematical analysis at the time of their creation? Is there any connection of such artistic creations with the scientific investigation of geometry of polyhedra at corresponding historical periods? What is the earliest historical period when such analysis could be performed in China? (2) Is it possible that some older "fullerene-like" artefacts can be found in medieval Chinese sculptures? What is the earliest historical period when such artefacts could have been created?

We know the answer to the first question: the mathematical analysis of such complicated polygonal structures could have been performed in China only after the scientific and pedagogical activity of the Jesuit priest Matteo Ricci (1552–1610) and his followers at the Imperial Court in Beijing. They introduced the geometry of polyhedra into China.

Wending Mei was the first Chinese mathematician who, in the early Qing period, did a thorough study on the Platonic solids in his book, "*The Jihe bubian*" (JHBB, Supplementary Notes on Geometry, 1692) [10], in which he intended to make a more complete study to extend the short descriptions of these five solids given previously by Jacobus Rho in "*Celiang Quanyi*" [11]. In addition to the extensive studies on geometric properties of the five Platonic solids in JHBB, Mei also described two Archimedean solids, cuboctahedron and icosidodecahedron (Figure 8a–b), quite carefully [12, 13]. There is a short supplementary note in JHBB that indicates four other Archimedean solids, truncated tetrahedron, truncated cube, truncated octahedron, rhombicuboctahedron, and one nonconvex stellated solid, stella octangula, proposed by his friend, Linzong Kong. It is worth noting that only a small part of the icosidodecahedron (Figure 8b) was shown in JHBB [12, 13]. The complete drawing of icosidodecahedron, particularly the superimposed figure with dodecahedron or icosahedron simultaneously, which would look exactly like pentakis-dodecahedron (C_{60}-dual, C_{60}^{*}) (see Appendix), is not shown in JHBB. Thus, it is hard to tell whether Mei knew the C_{60}^{*} structure or not.

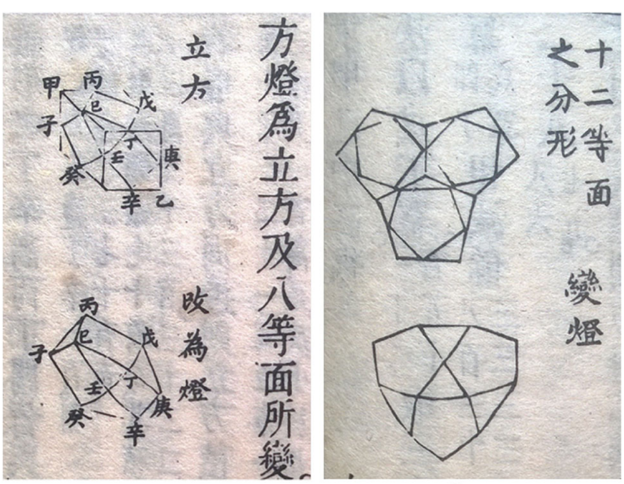


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Figure 8. Two Archimedean solids described in Wending Mei's "*Jihe bubian*": (a) Cuboctahedron, (b) Icosidodecahedron [10].

Appendix: Fullerenes and Their Duals

The C_{60} molecule, with carbon atoms at the 60 vertices of a truncated icosahedron (an Archimedean solid in the shape of a modern soccer ball) (Fig. 9a), discovered in 1985 [14], was named buckminsterfullerene, or buckyball, because of the connection between the C_{60} molecular structure and Buckminster Fuller's geodesic domes. The discovery of C_{60} was followed by the hypothesis of the existence of an entire family of carbon molecules, the fullerenes, in the shape of convex polyhedra with vertices of degree three and only pentagonal and hexagonal faces [15] that was afterward experimentally confirmed. Mathematicians often give their own definition [16]: a fullerene (C_V) is a polyhedron with V vertices of degree three and only pentagonal and hexagonal faces [17].

For any convex polyhedron with F faces, E edges, and V vertices, the Euler relation holds [18, 19]:

$$V - E + F = 2 \tag{1}$$

It is easy to show that the faces cannot all be hexagons. For fullerenes, where f_6 and f_5 are the numbers of hexagonal and pentagonal faces, respectively, it is almost as easy to show that $f_5 = 12$ and $V = 2(10 + f_6)$. Thus the number of pentagonal faces is always 12. The value of f_6 can be any number but 1 [20]. Accordingly, the smallest fullerene, C_{20}, has the shape of a Platonic dodecahedron, formed only by pentagons (Fig. 9b). The next fullerenes are C_{24}, C_{26}, C_{28}, …, C_{60}, C_{70}, $C_{2(10+h)}$ …

According to group theory, the C_{60} structure (truncated icosahedron) belongs to the same point group of symmetry as the Platonic icosahedron and dodecahedron, I_h. This

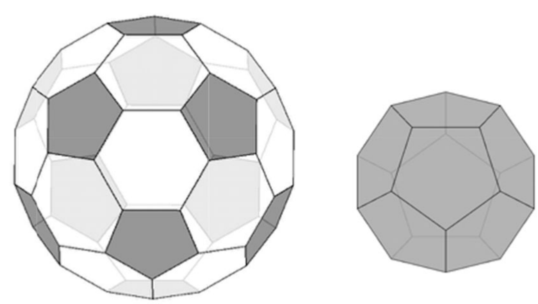

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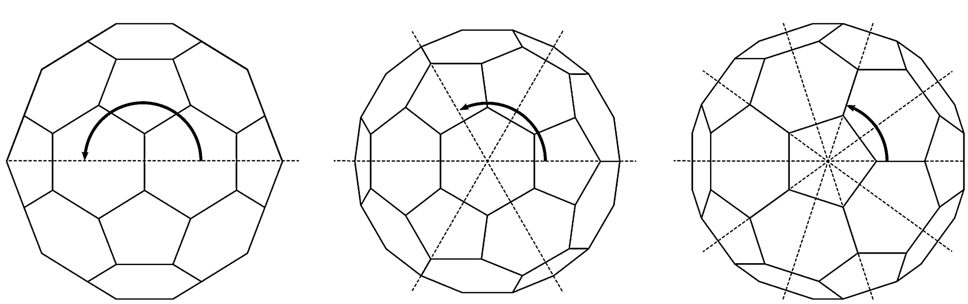
Figure 9. (a) C_{60}, (b) C_{20}.



a b c

Figure 10. Axes of rotational symmetry in truncated icosahedron: (a) twofold axes, (b) threefold axes, (c) fivefold axes.

Table 1. Characteristics of Platonic Solids

Polyhedron	Number of Edges per Each Face, m	Vertex Degree (Number of Edges that Connect in Each Vertex), n	Number of Faces, F	Number of Edges, E	Number of Vertices, V
Tetrahedron	3	3	4	6	4
Cube	4	3	6	12	8
Octahedron	3	4	8	12	6
Icosahedron	3	5	20	30	12
Dodecahedron	5	3	12	30	20

group of icosahedral symmetry includes the highest number of symmetry elements (120 elements) of all types: center of symmetry (inversion center), planes of mirror reflection (bilateral symmetry), and axes of two-, three-, and fivefold rotational symmetry (Fig. 10).

Some other fullerenes, although not all, belong to the I_h group as well as to another icosahedral group I (without the center of symmetry). No such symmetrical molecule was known before the discovery of C_{60}.

The number of vertices (carbon atoms in a molecule C_V) V in all icosahedral fullerenes can be described by very simple formulas:

$$V = 20T \tag{2}$$

and

$$T = a^2 + ab + b^2, \tag{3}$$

where a and b are integer numbers.

For fullerenes with I_h symmetry: $a = b \neq 0$ or $b = 0$.

For example, the smallest fullerene C_{20}, Platonic dodecahedron, is characterized by $(a, b) = (1, 0)$ and $T = 1$; C_{60}, truncated icosahedron by $(a, b) = (1, 1)$ and $T = 3$; C_{80} with I_h symmetry by $(a, b) = (2, 0)$ and $T = 4$.

Icosahedral fullerene polyhedra are commonly referred to as Goldberg polyhedra [21] and parameters (a, b) are considered as coordinates of the *Goldberg vector* that connects the two closest pentagons. The credit for the formal definition of this polyhedral family and the analysis with Eqs. (2–3) goes to American mathematician Michael Goldberg who published a pioneering paper in 1937 [22]. This paper received a lot of credit in the context of pioneering research of the structure of viruses [23].

In geometry, polyhedra are associated into pairs called *duals*, in which the vertices of one correspond to the faces of the other. This correspondence can be simply explained by the example of the five regular *Platonic* solids, four of which form two pairs of such duals (cube-octahedron and icosahedron-dodecahedron) (Table 1).

As one can see from Table 1, the numbers F and V are interchangeable for cube and octahedron (6,8) – (8,6) and for icosahedron and dodecahedron (20,12) – (12,20), respectively. The parameters m and n are also interchangeable for these pairs of polyhedra. Cube and octahedron are dual to each other; icosahedron is dual to dodecahedron (and vice versa), and tetrahedron is dual to itself.

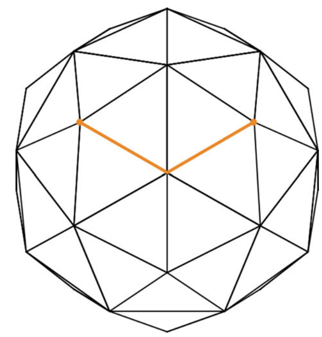


Figure 11. Pentakis-dodecahedron, dual of truncated icosahedron.

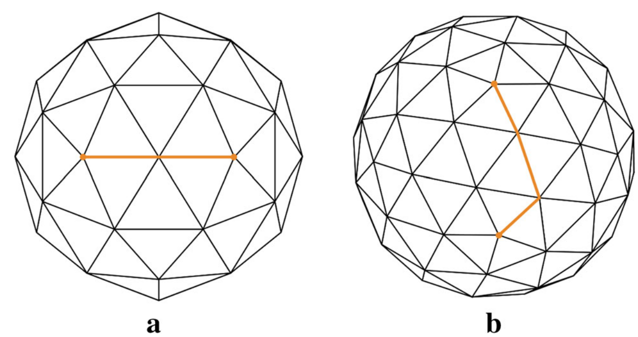


Figure 12. Duals of icosahedral fullerenes: (a) C^*_{80} [I_h; (a, b) = (2, 0)]; (b) C^*_{140} [I; (a, b) = (2, 1)].

An important property of a pair of dual polyhedra is that both possess the same symmetry and they must have the same number of edges.

The dual of an isogonal polyhedron, having equivalent vertices, is one that is isohedral, having equivalent faces.

Because fullerene polyhedra have 12 pentagonal and a various number of hexagonal faces and all their vertices have the degree of 3, fullerene duals have only triangular faces, 12 vertices with degree of 5 and various number of vertices with degree of 6. In the sequel, we denote a dual of fullerene C_V by C^*_V.

The smallest fullerene dual is the Platonic icosahedron (dual of the dodecahedron) with 20 triangular faces and 12 five-valent vertices. The dual of truncated icosahedron, C_{60}, is pentakis-dodecahedron C^*_{60} with 60 triangular faces, 12 five-degree vertices, and 20 six-degree vertices (Fig. 11).

Since every fullerene polyhedron and its dual share the same symmetry group, duals of icosahedral fullerenes also have icosahedral symmetry. The only difference is that Eq. (2) describes the number of faces in duals C^{*}_{20T}, where T again is equal to $a^2 + ab + b^2$. Now, the parameters a and b represent the coordinates of the vector joining the two closest vertices of degree five (Fig. 12).

Many of the famous geodesic domes and most of icosahedral viruses have the structure of dual icosahedral fullerenes [24].

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2. According to Jun Cheng's "Traditional Lion Art in China (中國傳統獅子工藝)," published by Beijing Arts and Crafts Publishing House (北京工藝美術出版社, 2013, Beijing), there are six pairs of bronze lion statues in the forbidden city, which are located in front of Taihe Gate (太和門), Qianqing Gate (乾清門), Yangxin Dian (養心殿), Changchun Palace (長春宮), Ningshou Gate (寧壽門), and Yangxing Dian (養性殿), respectively. Three male lions in front of Qianqing Gate, Ningshou Gate, and Yangxing Dian step on balls decorated with fullerene-like patterns; whereas the two balls under male lions in front of Yangxin Dian and Changchun Palace have triangular tiling patterns.
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4. The Qing Dynasty inherited the Forbidden City directly from the Ming Dynasty. Without the need to build a new palace, the focus thus moved to the construction of the Imperial Gardens. In the peak of the Qing dynasty (Qianlong's reign), more than 90 imperial gardens were built in the great Beijing area. The most famous gardens among them are Yuanmingyuan (Old Summer Palace, 圓明園) and Qingyiyuan (清漪園). Both Yuanmingyuan and Qingyiyuan were destroyed by Anglo-French armies during the Second Opium War in 1860. Between 1884 and 1895, during the reign of the Guanxu Emperor, Qingyiyuan was reconstructed and given its present-day name, "Yiheyuan" (New Summer Palace). There were at least two pairs of bronze lion statues in the Yuanmingyuan and one pair of bronze lion statues in the Qingyiyuan. The British army took one pair of bronze lion statues located in Yuanminyuan during the Second Opium War and returned them back to China more than 100 years later, in 1984. This pair is now located in the Diaoyutai State Guest House, Beijing. The other pair from Yuanmingyuan is now located in front of Paiyun Gate of Yiheyuan.
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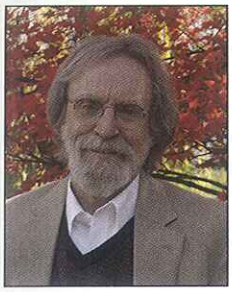
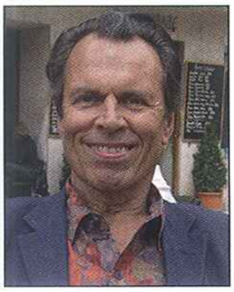
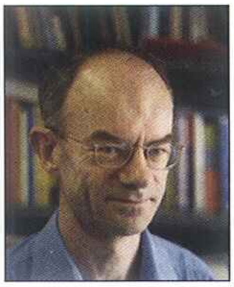
Contributors



WILLIAM DUNHAM, a historian of mathematics, has written four books and recently co-edited, with Don Albers and Jerry Alexanderson, *The G. H. Hardy Reader* (Cambridge, 2016), an anthology of writings by and about this eccentric British mathematician from a century ago. Dunham is also featured in the Teaching Company's DVD course, "Great Thinkers, Great Theorems." After retiring from a 22-year career at Muhlenberg College, Dunham adopted the philosophy, "Have Math History, Will Travel;" this has carried him to many visiting positions. Last year he was named a Research Associate in Mathematics at Bryn Mawr College.

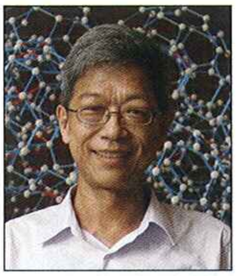


PAUL ERNEST is emeritus professor at Exeter University, UK, where he founded and directed the masters and doctoral programme in mathematics education. He has written extensively on aims, learning theory, epistemology, ethics and values and social constructivism. His books include *The Philosophy of Mathematics Education (Routledge 1991) and Social Constructivism as a Philosophy of Mathematics* (SUNY Press 1998). He founded and edits the *Philosophy of Mathematics Education Journal*, now in its 26th year, freely available online via <http://people.exeter.ac.uk/PErnest/>.



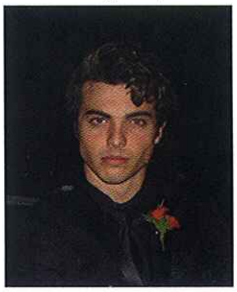
JAMES FRANKLIN earned a Ph.D. from the University of Warwick on algebraic groups. His book, *An Aristotelian Realist Philosophy of Mathematics*, has just been published. He has also written on the prehistory of probability, Australian philosophy, knowledge in science, and a textbook on proof in mathematics, and was awarded the 2005 Eureka Prize for Research in Ethics. His research interests include extreme risks, or how to deal with "data-free statistics": estimating the probability of events that haven't happened yet.

EUGENE A. KATZ Professor of Physics at the Ben-Gurion University of the Negev, received his Ph.D. in solid state physics at the Moscow Institute of Steel and Alloys. His research interests include photovoltaics based on non-traditional semiconductors (fullerenes, carbon nanotubes, conjugated polymers), photovoltaic characterization of concentrator solar cells, and solar synthesis of nanomaterials. In addition to his many research papers, he has published a book and articles for broader audiences on the history of science and fullerene-like structures in nanomaterials, living organisms and architecture.

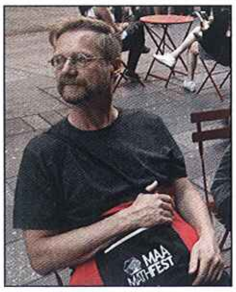


BIH-YAW JIN Professor of Chemistry at National Taiwan University, received his B.Sc. from that institution and his Ph. D. in Chemistry from Massachusetts Institute of Technology. His research interests include charge transport through a single molecular junction, classification of topologically nontrivial graphitic structures, electronic processes in organic and graphitic materials, and the connection between science and art through mathematical beading.

TANYA KHOVANOVA is a Lecturer at MIT and likes to entertain people with mathematics. She received her Ph.D. in Mathematics from the Moscow State University in 1988. Her current research interests lie in recreational mathematics including puzzles, magic tricks, combinatorics, number theory, geometry, and probability theory. Her website is located at tanyakhovanova.com, her highly popular math blog at blog.tanyakhovanova.com, and her Number Gossip website at numbergossip.com.



NICK DIACO participated in the MIT PRIMES-USA program (Program for Research in Mathematics, Engineering and Science for High School Students) in 2015 and was mentored by Dr. Tanya Khovanova. His mathematical interests include optimization and analytic number theory (although he does not plan on solving the Riemann hypothesis any time soon). He graduated from high school in June 2016, but he's chosen to take a gap year before attending college next year – he's currently in Taiwan working on research and development in the manufacturing sector. His hobbies include astrophotography, surfing, and wondering if anyone is actually going to read this whole thing.



JEAN-PIERRE MARQUIS is Vice-Dean at the Faculty of Arts and Sciences of the Université de Montréal and professor of philosophy . His areas of research are philosophy of mathematics, logic, epistemology and ontology with a special interest in contemporary mathematics, particularly category theory and categorical logic. His book "From a Geometrical Point of View: a study in the history and philosophy of category theory" was published by Springer in 2009. He is currently working on philosophical aspects of higher dimensional category theory and