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# Eigenforms, Discrete Processes and Quantum Processes

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**Abstract.** This essay is a discussion of the concept of eigenform, due to Heinz von Foerster, and its relationship with discrete physics and quantum mechanics. We interpret the square root of minus one as a simple oscillatory process - a clock, and as an eigenform. By taking a generalization of this identification of  $i$  as a clock and eigenform, we show how quantum mechanics emerges from discrete physics.

## 1. Introduction

This paper is an attempt to bridge two worlds, the world of second order cybernetics and the world of foundations of physics. We will begin with elementary ideas of recursion in a cybernetic context and use these ideas to discuss discrete dynamical systems and show how quantum mathematical structures emerge from discrete dynamics and the desire to describe such dynamics to include temporal observations that keep track of the ticks of the clock.

Our essay begins with explication of the notion of eigenform as pioneered by Heinz von Foerster in his papers [4, 5, 6, 7] and explored in papers of the author [11, 12, 22, 23]. In [5] The familiar objects of our existence can be seen as tokens for the behaviours of the organism, creating apparently stable forms. Such an attitude toward objects makes it impossible to discriminate between the object as an element of a world and the object as a token or symbol that is simultaneously a process.

The notion of an eigenform is inextricably linked with second order cybernetics. One starts on the road to such a concept as soon as one begins to consider a pattern of patterns, the form of form or the cybernetics of cybernetics. Such concepts appear to loop around upon themselves, and at the same time they lead outward to new points of view. Such circularities suggest a possibility of transcending the boundaries of a system from within. When a circular concept is called into being, the boundaries turn inside out.

An object, in itself, is a symbolic entity, participating in a network of interactions, taking on its apparent solidity and stability from these interactions. We ourselves are such objects, we as human beings are “signs for ourselves”, a concept originally due to the American philosopher C. S. Peirce [10]. Eigenforms are mathematical companions to Peirce’s work.

In an observing system, what is observed is not distinct from the system itself, nor can one make a separation between the observer and the observed. The observer and the observed stand together in a coalescence of perception. From the stance of the observing system all objects are

non-local, depending upon the presence of the system as a whole. It is within that paradigm that these models begin to live, act and enter into conversation with us.

The central metaphor of this paper is the temporal nexus where time is implicit, and time is explicit and keeping time. In the nexus there is neither form nor sign, motion nor time. Time, the measurement of time and time's indication all emerge at once from the nexus in the form of action that is embodied in it. The metaphor suggests that it is no accident that deeper physical reality is revealed when mere numerical time  $t$  is replaced by the time of the nexus  $it$ . The time of the nexus is at once flowing, beyond motion, an eigenform, a geometric operator and a discrete dynamics counting below where counting cannot go.

Taking the point of view described above we reexamine the square root of minus one starting with the simple oscillatory processes that ensue from iterating the operator  $R(x) = -1/x$  whose fixed points are  $i$  and  $-i$ . By looking at  $i$  in this way one finds the essential patterns of other discrete processes and we detail how this works in Section 7. Then by weaving the temporal process for  $i$  with the general discrete process we find, in a natural way, that discrete processes are described by the Heisenberg commutator

$$[p, q] = i\hbar.$$

This paper is a sketch of this point of view.

## 2. Objects as Tokens for Eigenbehaviours

In his paper "Objects as Tokens for Eigenbehaviours" [5] von Foerster suggests that we think seriously about the mathematical structure behind the constructivist doctrine that perceived worlds are worlds created by the observer. At first glance such a statement appears to be nothing more than solipsism. At second glance, the statement appears to be a tautology, for who else can create the rich subjectivity of the immediate impression of the senses? In that paper he suggests that the familiar objects of our experience are the fixed points of operators. These operators are the structure of our perception. To the extent that the operators are shared, there is no solipsism in this point of view. It is the beginning of a mathematics of second order cybernetics.

Consider the relationship between an observer  $O$  and an "object"  $A$ . "The object remains in constant form with respect to the observer". This constancy of form does not preclude motion or change of shape. Form is more malleable than the geometry of Euclid. In fact, ultimately the form of an "object" is the form of the distinction that "it" makes in the space of our perception. In any attempt to speak absolutely about the nature of form we take the form of distinction for the form. (paraphrasing Spencer-Brown [3]). It is the form of distinction that remains constant and produces an apparent object for the observer. How can you write an equation for this? We write

$$O(A) = A.$$

The object  $A$  is a fixed point for the observer  $O$ . The object is an eigenform. We must emphasize that this is a most schematic description of the condition of the observer in relation to an object  $A$ . We record only that the observer as an actor (operator) manages to leave the (form of) the object unchanged. This can be a recognition of symmetry, but it also can be a description of how the observer, searching for an object, makes that object up (like a good fairy tale) from the very ingredients that are the observer herself.

And what about this matter of the object as a token for eigenbehaviour? This is the crucial step. We forget about the object and focus on the observer. We attempt to "solve" the equation  $O(A) = A$  with  $A$  as the unknown. Not only do we admit that the "inner" structure of the object is unknown, we adhere to whatever knowledge we have. We can start anew from the

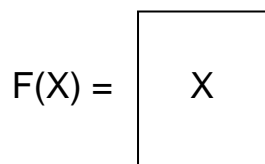


Figure 1.

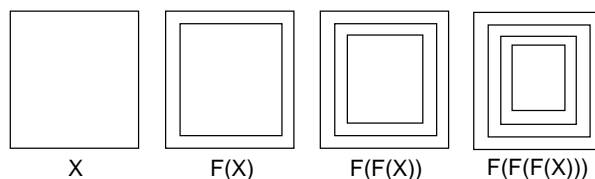


Figure 2.

dictum that the perceiver and the perceived arise together in the condition of observation. This is mutuality. Neither perceiver nor the perceived have priority over the other. A distinction has emerged and with it a world with an observer and an observed. The distinction is itself an eigenform.

### 3. The Eigenform Model

We have seen how the concept of an object has evolved. The notion of a fixed object has become the notion of a process that produces the apparent stability of an object. This process can be simplified in modelling to become a recursive process where a rule or rules are applied time and time again. The resulting object is the fixed point or eigenform of the process, and the process itself is the eigenbehaviour.

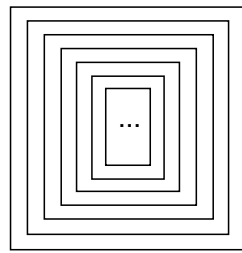
In this way we have a model for thinking about object as token for eigenbehaviour. This model examines the result of a simple recursive process carried to its limit. For example, suppose that we have a recursion as shown in Figure 1.

Each step in the process encloses the results of the previous step within a box. In Figure 2 we illustrate the first few steps of the process applied to an empty box  $X$ :

If we continue this process, then successive nests of boxes resemble one another, and in the limit of infinitely many boxes, we find that the infinite nest of boxes is invariant under the addition of one more surrounding box. Hence this infinite nest of boxes is a fixed point for the recursion. In other words, if  $J$  denotes the infinite nest of boxes, then  $J = F(J)$  as shown in Figure 3. This equation is a description of a state of affairs. The form of an infinite nest of boxes is invariant under the operation of adding one more surrounding box.

In the process of observation, we interact with ourselves and with the world to produce stabilities that become the objects of our perception. These objects, like the infinite nest of boxes, often go beyond the specific properties of the world in which we operate. We make an imaginative leap to complete such objects to become tokens for eigenbehaviours. It is impossible to make an infinite nest of boxes. We do not make it. We imagine it. And in imagining that infinite nest of boxes, we arrive at the eigenform.

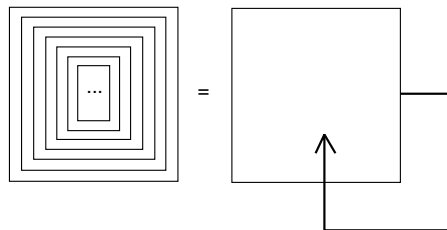
Sometimes one stylizes the structure by indicating where the eigenform  $X$  reenters its own indicational space by an arrow or other graphical device. See Figure 4 for this stylization for the case of the nested boxes.



$$J = F(F(F(F(F(\dots))))))$$

$$J = F(J)$$

**Figure 3.**



**Figure 4.**

An object is an amphibian between the symbolic and imaginary world of the mind and the complex world of personal experience. The object, when viewed as process, is a dialogue between these worlds. The object when seen as a sign for itself, or in and of itself, is imaginary. The perceiving mind is itself an eigenform of its own perception.

#### 4. The Square Root of Minus One is a Clock

The purpose of this section is to place  $i$ , the square root of minus one, and its algebra in the context of eigenform and reflexivity. We begin by starting with a simple periodic process that is associated directly with the classical attempt to solve for  $i$  as a solution to a quadratic equation. We take the point of view that solving  $x^2 = ax + b$  is the same (when  $x \neq 0$ ) as solving

$$x = a + b/x,$$

and hence is a matter of finding a fixed point. In the case of  $i$  we have

$$x^2 = -1$$

and so desire a fixed point

$$x = -1/x.$$

There are no real numbers that are fixed points for this operator and so we consider the oscillatory process generated by

$$R(x) = -1/x.$$

The fixed point would satisfy

$$i = -1/i$$

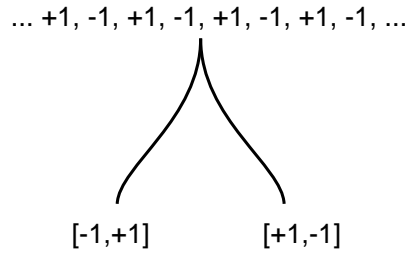


Figure 5.

and multiplying, we get that

$$ii = -1.$$

On the other hand the iteration of  $R$  yields

$$1, R(1) = -1, R(R(1)) = +1, R(R(R(1))) = -1, +1, -1, +1, -1, \dots$$

The square root of minus one is a perfect example of an eigenform that occurs in a new and wider domain than the original context in which its recursive process arose. The process has no fixed point in the original domain.

Looking at the oscillation between  $+1$  and  $-1$ , we see that there are naturally two phase-shifted viewpoints. We denote these two views of the oscillation by  $[+1, -1]$  and  $[-1, +1]$ . These viewpoints correspond to whether one regards the oscillation at time zero as starting with  $+1$  or with  $-1$ . See Figure 5.

We shall let  $I\{+1, -1\}$  stand for an undisclosed alternation or ambiguity between  $+1$  and  $-1$  and call  $I\{+1, -1\}$  an iterant. There are two iterant views:  $[+1, -1]$  and  $[-1, +1]$ .

Given an iterant  $[a, b]$ , we can think of  $[b, a]$  as the same process with a shift of one time step. These two iterant views, seen as points of view of an alternating process, will become the square roots of negative unity,  $i$  and  $-i$ .

We introduce a temporal shift operator  $\eta$  such that

$$[a, b]\eta = \eta[b, a]$$

and

$$\eta\eta = 1$$

for any iterant  $[a, b]$ , so that concatenated observations can include a time step of one-half period of the process

$$\dots abababab \dots$$

We combine iterant views term-by-term as in

$$[a, b][c, d] = [ac, bd].$$

We now define  $i$  by the equation

$$i = [1, -1]\eta.$$

This makes  $i$  both a value and an operator that takes into account a step in time.

We calculate

$$ii = [1, -1]\eta[1, -1]\eta = [1, -1][-1, 1]\eta\eta = [-1, -1] = -1.$$

Thus we have constructed the square root of minus one by using an iterant viewpoint. In this view  $i$  represents a discrete oscillating temporal process and it is an eigenform for  $R(x) = -1/x$ , participating in the algebraic structure of the complex numbers. In fact the corresponding algebra structure of linear combinations  $[a, b] + [c, d]\eta$  is isomorphic with  $2 \times 2$  matrix algebra and iterants can be used to construct  $n \times n$  matrix algebra. We treat this generalization elsewhere [20, 22, 23].

**The Temporal Nexus.** *We take as a matter of principle that the usual real variable  $t$  for time is better represented as  $it$  so that time is seen to be a process, an observation and a magnitude all at once.* This principle of “imaginary time” is justified by the eigenform approach to the structure of time and the structure of the square root of minus one.

As an example of the use of the Temporal Nexus, consider the expression  $x^2 + y^2 + z^2 + t^2$ , the square of the Euclidean distance of a point  $(x, y, z, t)$  from the origin in Euclidean four-dimensional space. Now replace  $t$  by  $it$ , and find

$$x^2 + y^2 + z^2 + (it)^2 = x^2 + y^2 + z^2 - t^2,$$

the squared distance in hyperbolic metric for special relativity. By replacing  $t$  by its process operator value  $it$  we make the transition to the physical mathematics of special relativity.

## 5. Quantum Physics, Eigenvalue and Eigenform

In quantum modelling [11], the state of a physical system is represented by a vector in a Hilbert space. As time goes on the vector undergoes a unitary evolution in the Hilbert space. Observable quantities correspond to Hermitian operators  $H$  and vectors  $v$  that have the property that the application of  $H$  to  $v$  results in a new vector that is a multiple of  $v$  by a real factor  $\lambda$ . Thus

$$Hv = \lambda v.$$

One says that  $v$  is an eigenvector for the operator  $H$ , and that  $\lambda$  is the eigenvalue.

The theory of eigenforms is a generalization that shifts eigenvectors to eigenforms. One can speculate that there might be a generalization of quantum mechanics where the eigenvectors are replaced by eigenforms and the Hilbert space involves superpositions of forms. In fact, we have formalized some of these ideas in our theory of quantum knots, but will not discuss that avenue in this paper. Here we are concerned with eigenforms in relation to discrete processes. The simplest case of this way of thinking is the treatment of  $i$  that we have used, making it into a discrete clock and making it possible to view  $i$  itself as an eigenform that represents this elementary discrete process. In the next sections we will analyse discrete processes in more generality and show how they partake of the patterns of elementary quantum physics.

## 6. The Wave Function in Quantum Mechanics and The Square Root of Minus One

One can regard a wave function such as  $\psi(x, t) = \exp(i(kx - wt))$  as containing a micro-oscillatory system with the special synchronizations of the iterant view  $i = [+1, -1]\eta$ . It is these synchronizations that make the big eigenform of the exponential work correctly with respect to differentiation, allowing it to create the appearance of rotational behaviour, wave behaviour and the semblance of the continuum. In other words, we are suggesting that once can take a temporal view of the well-known equation of Euler:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

by regarding the  $i$  in this equation as an iterant, as discrete oscillation between  $-1$  and  $+1$ . One can blend the classical geometrical view of the complex numbers with the iterant view by thinking of a point that orbits the origin of the complex plane, intersecting the real axis periodically and producing, in the real axis, a periodic oscillation in relation to its orbital movement in the two dimensional space. The special synchronization is the algebra of the time shift embodied in

$$\eta\eta = 1$$

and

$$[a, b]\eta = \eta[b, a]$$

that makes the algebra of  $i = [1, -1]\eta$  imply that  $i^2 = -1$ . This interpretation does not change the formalism of these complex-valued functions, but it does change one's point of view and in the next section we will show how the properties of  $i$  as a discrete dynamical system are found in any such system.

## 7. Time Series and Discrete Physics

We have just reformulated the complex numbers and expanded the context of matrix algebra to an interpretation of  $i$  as an oscillatory process and matrix elements as combined spatial and temporal oscillatory processes (in the sense that  $[a, b]$  is not affected in its order by a time step, while  $[a, b]\eta$  includes the time dynamic in its interactive capability, and  $2 \times 2$  matrix algebra is the algebra of iterant views  $[a, b] + [c, d]\eta$ ).

We now consider elementary discrete physics in one dimension. Consider a time series of positions

$$x(t) : t = 0, \Delta t, 2\Delta t, 3\Delta t, \dots$$

We can define the velocity  $v(t)$  by the formula

$$v(t) = (x(t + \Delta t) - x(t))/\Delta t = Dx(t)$$

where  $D$  denotes this discrete derivative. In order to obtain  $v(t)$  we need at least one tick  $\Delta t$  of the discrete clock. Just as in the iterant algebra, we need a time-shift operator to handle the fact that once we have observed  $v(t)$ , the time has moved up by one tick.

**We adjust the discrete derivative.** We shall add an operator  $J$  that in this context accomplishes the time shift:

$$x(t)J = Jx(t + \Delta t).$$

We then redefine the derivative to include this shift:

$$Dx(t) = J(x(t + \Delta t) - x(t))/\Delta t.$$

This readjustment of the derivative rewrites it so that the temporal properties of successive observations are handled automatically.

**Discrete observations do not commute.** Let  $A$  and  $B$  denote quantities that we wish to observe in the discrete system. Let  $AB$  denote the result of first observing  $B$  and then observing  $A$ . The result of this definition is that a successive observation of the form  $x(Dx)$  is distinct from an observation of the form  $(Dx)x$ . In the first case, we first observe the velocity at time  $t$ , and then  $x$  is measured at  $t + \Delta t$ . In the second case, we measure  $x$  at  $t$  and then measure the velocity.



We measure the difference between these two results by taking a commutator

$$[A, B] = AB - BA$$

and we get the following computations where we write  $\Delta x = x(t + \Delta t) - x(t)$ .

$$x(Dx) = x(t)J(x(t + \Delta t) - x(t)) = Jx(t + \Delta t)(x(t + \Delta t) - x(t)).$$

$$(Dx)x = J(x(t + \Delta t) - x(t))x(t).$$

$$[x, Dx] = x(Dx) - (Dx)x = (J/\Delta t)(x(t + \Delta t) - x(t))^2 = J(\Delta x)^2/\Delta t$$

This final result is worth recording:

$$[x, Dx] = J(\Delta x)^2/\Delta t.$$

From this result we see that the commutator of  $x$  and  $Dx$  will be constant if  $(\Delta x)^2/\Delta t = K$  is a constant. For a given time-step, this means that

$$(\Delta x)^2 = K\Delta t$$

so that

$$\Delta x = \pm\sqrt{(K\Delta t)}$$

This is a Brownian process with diffusion constant equal to  $K$ .

Thus we arrive at the result that any discrete process viewed in this framework of discrete observation has the basic commutator

$$[x, Dx] = J(\Delta x)^2/\Delta t,$$

generalizing a Brownian process and containing the factor  $(\Delta x)^2/\Delta t$  that corresponds to the classical diffusion constant. It is worth noting that the adjustment that we have made to the discrete derivative makes it into a commutator as follows:

$$Dx(t) = J(x(t + \Delta t) - x(t))/\Delta t = (x(t)J - Jx(t))\Delta t = [x(t), J]/\Delta t.$$

By replacing discrete derivatives by commutators we can express discrete physics in many variables in a context of non-commutative algebra (non-commutative worlds as in [20]).

In the next section we use the temporal nexus (the square root of minus one as a clock) and rewrite these commutators to match quantum mechanics.

## 8. Epilogue and Simplicity

Finally, we arrive at the simplest place. Time and the square root of minus one are inseparable in the temporal nexus. The square root of minus one is a symbol and algebraic operator for the simplest oscillatory process. As a symbolic form,  $i$  is an eigenform satisfying the equation

$$i = -1/i.$$

One does not have an increment of time all alone as in classical  $t$ . One has  $it$ , a combination of an interval and the elemental dynamic that is time. With this understanding, we can return to the commutator for a discrete process and use  $it$  for the temporal increment.

We found that discrete observation led to the commutator equation

$$[x, Dx] = J(\Delta x)^2/\Delta t$$

which we will simplify to

$$[q, p/m] = (\Delta x)^2/\Delta t.$$

taking  $q$  for the position  $x$  and  $p/m$  for velocity, the time derivative of position and ignoring the time shifting operator on the right hand side of the equation.

Understanding that  $\Delta t$  should be replaced by  $i\Delta t$ , and that, by comparison with the physics of a process at the Planck scale one can take

$$(\Delta x)^2/\Delta t = \hbar/m,$$

we have

$$[q, p/m] = (\Delta x)^2/i\Delta t = -i\hbar/m,$$

whence

$$[p, q] = i\hbar,$$

and we have arrived at Heisenberg's fundamental relationship between position and momentum. This mode of arrival is predicated on the recognition that only  $it$  represents a true interval of time. In the notion of time there is an inherent clock or an inherent shift of phase that is making a synchrony in our ability to observe, a precise dynamic beneath the apparent dynamic of the observed process. Once this substitution is made, once the correct imaginary value is placed in the temporal circuit, the patterns of quantum mechanics appear. In this way, quantum mechanics can be seen to emerge from the discrete.

The problem that we have examined in this paper is the problem to understand the nature of quantum mechanics. In fact, we hope that the problem is seen to disappear the more we enter into the present viewpoint. A viewpoint is only on the periphery. The iterant from which the viewpoint emerges is in a superposition of indistinguishables, and can only be approached by varying the viewpoint until one is released from the particularities that a point of view contains.

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