Calculus, Relativity and Non-Commutative Worlds

Louis H. Kauffman
Department of Mathematics, Statistics and Computer Science
University of Illinois at Chicago
851 South Morgan Street
Chicago, IL, 60607-7045

Abstract. This paper shows how gauge theoretic structures arise naturally in a non-commutative calculus.

Keywords. derivative, derivation, commutator, curvature, parallel translation, covariant derivative, connection, Levi-Civita connection, gauge connection, holonomy, discrete derivative, quantum theory, quantum gravity.

AMS Classification. 53Z05.

1 Introduction to Non-Commutative Worlds

Calculus was formulated in a commutative framework by Newton and his successors. We are all familiar with the limit definitions for derivatives and we take for granted that classical mechanics is formulated in this framework. The advent of quantum mechanics brought with it formulations of physical theory that are deeply related to non-commutativity. Heisenberg invented a calculus of quantum quantities that did not commute with one another and obeyed specific identities such as the famous

$$QP - PQ = \hbar i.$$

Schrödinger gave a formulation of quantum mechanics using standard partial differential equations and then discovered that the operators $Q = x$ and $P = -i\hbar\partial/\partial x$ obeyed exactly Heisenberg's relations and gave the translation between his work and the Heisenberg viewpoint. Dirac discovered a key to quantization via the replacement of the Poisson bracket of Hamiltonian mechanics with the commutators of quantum operators. The curvatures in differential geometry and general relativity were seen, through the work of Weyl and others to correspond to measurements in differences in parallel translation and correspondingly, in the commutators of covariant derivatives. This emphasis on parallel translation led to the generalizations of differential geometry fundamental to gauge theory. Gauge theory began, with the work of Hermann Weyl [24] as a generalization of differential geometry where lengths as well as angles were dependent upon the choice of paths. Weyl saw how to incorporate electromagnetism into general relativity along these lines. Later the ideas of Weyl were adopted in the context of quantum mechanics and became the basis for the understanding of nuclear forces. Throughout all this development, the underlying calculus remained in the commutative and continuum realms.

In this paper we shall begin by formulating calculus in non-commutative domains. Our constructions are motivated by discrete calculus. Discrete calculus is naturally embedded in a non-commutative context, and there can be adjusted so that it satisfies the derivative of a product in the form of the Leibniz rule $D(FG) = D(F)G + FD(G)$. We explain this point in Section 6, an Appendix on discrete calculus.

If we take commutators $[A, B] = AB - BA$ in an abstract algebra and define $DA = [A, J]$ for a fixed element J, then D acts like a derivative in the sense that $D(AB) = D(A)B + AD(B)$ (the Leibniz rule). As soon as we have calculus in such a framework, concepts of geometry are immediately available. For example, if we have two derivatives $\nabla_J A = [A, J]$ and $\nabla_K A = [A, K]$, then we can consider the commutator of these derivatives

$$[\nabla_J, \nabla_K]A = \nabla_J \nabla_K A - \nabla_K \nabla_J A = [[J, K], A].$$

(The verification of this last inequality is an exercise for the reader.) We can regard $R_{JK} = [J, K]$ as the *curvature* associated with ∇_J and ∇_K. Note that the commutator of the derivations ∇_J and ∇_K. is represented by $R_{JK} = [J, K]$ so that when the representatives J and K for two derivations commute, then the derivations themselves commute and the curvature vanishes. The non-commutation of derivations corresponds to curvature in geometry. We shall see that the emergence of curvature in this context is the formal analog of the curvature of a gauge connection.

Aspects of gauge theory, Hamiltonian mechanics and quantum mechanics arise naturally in the mathematics of a non-commutative framework for calculus and differential geometry. This paper consists in 7 sections including the introduction. Section 2 outlines the general properties of calculus in a non-commutative domain, where derivatives are all represented by commutators. This includes a special element H such that the total time derivative $\dot{F} = [F, H]$ for any F in the non-commutative domain. We show how the formal analog of Hamilton's equations arises naturally in a flat coordinate system, and we show how Schrödinger's equation arises from the time derivative made appropriately complex. Section 3 explores the consequences of defining dynamics in the form

$$\dot{X}_i = dX_i/dt = \mathcal{G}_i$$

where $\{\mathcal{G}_1, \cdots, \mathcal{G}_d\}$ is a collection of elements of the non-commutative domain $\mathcal{N}$. Write $\mathcal{G}_i$ relative to flat coordinates via $\mathcal{G}_i = P_i - A_i$. This is a definition of A_i with $\partial_i F = \partial F/\partial X_i = [F, P_i]$. The formalism of gauge theory appears naturally via the curvature of ∇_i with $\nabla_i(F) = [F, P_i - A_i]$ that is given by the formula

$$R_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j].$$

With $\dot{X}_i = \mathcal{G}_i = P_i - A_i$, we let $g_{ij} = [X_i, \dot{X}_j]$ and show that this is a natural choice for a generalized metric. In particular, we show that for a quadratic Hamiltonian with these metric coefficients, the formula $g_{ij} = [X_i, \dot{X}_j]$ is a consequence. Furthermore, we show that for any F,

$$\dot{F} = \frac{1}{2}(\dot{X}_i \partial_i(F) + \partial_i(F)\dot{X}_i).$$

This means that with this choice of Hamiltonian the non-commutative world satisfies a correct analog to standard time derivative formula in commutative contexts. Higher order constraints of this sort are considered in [17]. Here we work at the level of this first constraint. We show how a covariant version of the Levi-Civita connection arises naturally in this commutator calculus. This connection satisfies the formula

$$\Gamma_{kij} + \Gamma_{ikj} = \nabla_j g_{ik} = \partial_j g_{ik} + [g_{ik}, A_j].$$

and so is exactly a generalization of the connection defined by Hermann Weyl in his original gauge theory [24]. In the non-commutative world $\mathcal{N}$ the metric indeed has a wider variability than the classical metric and its angular holonomy. Weyl's idea was to work with such a wider variability of the metric. The present formalism provides a new context for Weyl's original idea. The rest of Section 3 discusses the possibility of using this Levi-Connection to formulate a corresponding covariant derivative, Einstein tensor and general relativity. A direct formulation gives a curvature tensor without the usual symmetries (due to the presence of the covariant derivatives ∇_i) and so the corresponding Einstein tensor will not have vanishing divergence. The same issue arises in the original Weyl theory. Here is becomes a problem for further exploration. Section 4 recapitulates our constructs of the Levi-Civita connection in an index-free fashion. Section 5 is an appendix on the structure of the Einstein tensor and how the Bianchi identity can be seen from the Jacobi idenity in a non-commutative world. Section 6 is an appendix on how discrete calculus embeds in non-commutative calculus. Section 7 is an appendix on the Weyl 1-form leading to electromagnetism and a reminder of how this formalism is generalized to gauge theories, loop quantum gravity and low-dimensional topology. We end with a question about the Ashtekar variables, loop quantum gravity and their relationship with non-commutative worlds. This question will be taken up in a sequel to the present paper.

Remark. In our papers [17, 18] we have examined relationships of non-commutative worlds with general relativity in relation to higher order constraints. The results of this paper will be compared and combined with that previous work. Here we have not mentioned the relationships with the Feynman-Dyson derivation of electromagnetism from commutator calculus [3, 6, 19, 21] that were the initial impetus for this work [8]. It was that impetus that led to our work on non-commutative worlds including the following references [9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. In this paper we return to this subject in a new way.

2 Calculus in Non-Commutative Worlds

Our constructions are performed in a Lie algebra $\mathcal{N}$. One may take $\mathcal{N}$ to be a specific matrix Lie algebra, an abstract Lie algebra or as an associative algebra closed under the operation of commutation. That is, if A and B are elements of $\mathcal{N}$, then $[A, B] = AB - BA$ is also an element of $\mathcal{N}$. If $\mathcal{N}$ is taken to be an abstract Lie algebra, then it is convenient to use the universal enveloping algebra so that the Lie product can be expressed as a commutator.

On $\mathcal{N}$, a variant of calculus is built by defining derivations as commutators (or more generally as Lie products). For a fixed N in $\mathcal{N}$ one defines

$$\nabla_N : \mathcal{N} \longrightarrow \mathcal{N}$$

by the formula

$$\nabla_N F = [F, N] = FN - NF.$$

∇_N is a derivation satisfying the Leibniz rule.

$$\nabla_N(FG) = \nabla_N(F)G + F\nabla_N(G).$$

In $\mathcal{N}$ there are as many derivations as there are elements of the algebra, and these derivations behave quite wildly with respect to one another. If one takes. as in the introduction to the present paper, the concept of *curvature* as the non-commutation of derivations, then $\mathcal{N}$ is a highly curved world indeed. Within $\mathcal{N}$ one can build a tame world of derivations that mimics the behaviour of flat coordinates in Euclidean space. In order to have flat coordinates, we need that the derivations for those coordinate directions commute with one another. This, in turn, is implied by the commuting of the representatives for those derivatons. That is, suppose that X and Y are coordinates and that P_X and P_Y represent derivatives in thes directions so that one writes

$$\partial_X F = [F, P_X]$$

and

$$\partial_Y F = [F, P_Y].$$

Then

$$\partial_X \partial_Y = \partial_Y \partial_X$$

when $[P_X, P_Y] = 0$, as we have seen in the introduction. A flat coordinate system corresponds to a collection of commutator equations.

We take a collection of special elements $X_1, X_2, \cdots, X_d$ to represent coordinates. With $d = 3$ these can be the familiar pattern of three dimensional spatial coordinates. With $d = 4$ we can take X_4 to represent time, if this is desired. There is no a priori restriction on the value of d. As *flat coordinates* the X_i satisfy the commutator equations below with the P_j chosen to represent differentiation with respect to X_j.:

$$[X_i, X_j] = 0$$

$$[P_i, P_j] = 0$$

$$[X_i, P_j] = \delta_{ij}.$$

Derivatives are represented by commutators.

$$\partial_i F = \partial F/\partial X_i = [F, P_i],$$

$$\hat{\partial}_i F = \partial F/\partial P_i = [X_i, F].$$

The time derivative is represented by commutation with a special element H of the algebra:

$$dF/dt = [F, H].$$

The element H corresponds to the Hamiltonian in classical physics or the Hamiltonian operator in quantum physics. In the abstract world we are constructing, it is neither of these, but can be compared and represented by the classical or quantum Hamiltonians. For quantum mechanics, we take $i\hbar dA/dt = [A, H]$.

These non-commutative coordinates are the simplest flat set of coordinates for description of temporal phenomena in a non-commutative world. Note that *Hamilton's equations are a consequence of these definitions.* The very short proof of this fact is given below.

Hamilton's Equations.

$$dP_i/dt = [P_i, H] = -[H, P_i] = -\partial H/\partial X_i$$

$$dX_i/dt = [X_i, H] = \partial H/\partial P_i.$$

These are exactly Hamilton's equations of motion. The pattern of Hamilton's equations is built into the system.

3 Dynamics, Gauge Theory and the Weyl Theory.

One can take the general dynamical equation in the form

$$\dot{X}_i = dX_i/dt = \mathcal{G}_i$$

where $\{\mathcal{G}_1, \cdots, \mathcal{G}_d\}$ is a collection of elements of $\mathcal{N}$. Write $\mathcal{G}_i$ relative to the flat coordinates via $\mathcal{G}_i = P_i - A_i$. This is a definition of A_i and $\partial_i F = \partial F/\partial X_i = [F, P_i]$. The formalism of gauge theory appears naturally. In particular, defining derivations corresponding to the $\mathcal{G}_i$ by the formulas

$$\nabla_i(F) = [F, \mathcal{G}_i],$$

then one has the curvatures (as commutators of derivations)

$$[\nabla_i, \nabla_j]F = [[\mathcal{G}_i, \mathcal{G}_j], F]$$

(by the formula in the introduction to this paper)

$$= [[P_i - A_i, P_j - A_j], F]$$

$$= [[P_i, P_j] - [P_i, A_j] - [A_i, P_j] + [A_i, A_j], F]$$
$$= [\partial_i A_j - \partial_j A_i + [A_i, A_j], F].$$

Thus the curvature is given by the formula

$$R_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j].$$

We see that our curvature formula is the well-known formula for the curvature of a gauge connection when A_i is interpreted or represeented as such a connection. We shall now see that aspects of geometry arise naturally in this context, including the Levi-Civita connection (which is seen as a consequence of the Jacobi identity in an appropriate non-commutative world).

With $\dot{X}_i = P_i - A_i$, the commutator $[X_i, \dot{X}_j]$ takes the form

$$g_{ij} = [X_i, \dot{X}_j] = [X_i, P_j - A_j] = [X_i, P_j] - [X_i, A_j] = \delta_{ij} - [X_i, A_j].$$

Thus we see that the "gauge field" A_j provides the deviation from the Kronecker delta in this commutator. We have $[\dot{X}_i, \dot{X}_j] = R_{ij}$, so that these commutators represent the curvature.

Before proceeding further it is necessary to explain why $g_{ij} = [X_i, \dot{X}_j]$ is a correct analogue to the metric in the classical physical situation. In the classical picture we have $ds^2 = g_{ij} dx_i dx_j$ representing the metric. Hence $ds^2/dt^2 = g_{ij}(dx_i/dt)(dx_j/dt) = g_{ij} p_i p_j$ and so (with m = 1) the Hamiltonian is $H = \frac{1}{2} g_{ij} p_i p_j$. By convention we sum over repeated indices.

For the commutator $[X_i, \dot{X}_j] = g_{ij}$, this equation is a consequence of the right choice of Hamiltonian. In a given non-commutative world, we choose an H in the algebra to represent the total time derivative so that $\dot{F} = [F, H]$ for any F. Suppose we have elements g_{ij} such that

$$[g_{ij}, X_k] = 0,$$

$$[g_{ij}, P_k] = 0$$

and

$$g_{ij} = g_{ji}.$$

We choose

$$H = (g_{ij} P_i P_j + P_i P_j g_{ij})/4 = \frac{1}{2} g_{ij} P_i P_j.$$

This is the non-commutative analog of the classical $H = \frac{1}{2} g_{ij} p_i p_j$. We now show that this choice of Hamiltonian implies that $[X_i, \dot{X}_j] = g_{ij}$.

Lemma. Let g_{ij} be given such that $[g_{ij}, X_k] = 0$ and $g_{ij} = g_{ji}$. Define

$$H = \frac{1}{2} g_{ij} P_i P_j$$

(where we sum over the repeated indices) and

$$\dot{F} = [F, H].$$

Then

$$[X_i, \dot{X}_j] = g_{ij}.$$

Proof. Consider

$$[X_k, g_{ij}P_iP_j] = g_{ij}[X_k, P_iP_j]$$
$$= g_{ij}([X_k, P_i]P_j + P_i[X_k, P_j])$$
$$= g_{ij}(\delta_{ki}P_j + P_i\delta_{kj}) = g_{kj}P_j + g_{ik}P_i$$
$$= 2g_{kj}P_j.$$

Thus we have shown that

$$\dot{X}_k = g_{kj}P_j.$$

Then

$$[X_r, \dot{X}_k] = [X_r, g_{kj}P_j] = g_{kj}[X_r, P_j] = g_{kj}\delta_{rj} = g_{kr} = g_{rk}.$$

This completes the proof. □

We can further remark that

Lemma. With the same hypotheses as the previous Lemma and with F any element of the given non-commutative world $\mathcal{N}$, we have the formula

$$\dot{F} = \frac{1}{2}(\dot{X}_i\partial_i(F) + \partial_i(F)\dot{X}_i).$$

Proof. Note from the previous Lemma that $\dot{X}_k = g_{kj}P_j$.

$$\dot{F} = [F, H] = [F, \frac{1}{2}g_{ij}P_iP_j] = \frac{1}{2}g_{ij}[F, P_iP_j]$$
$$= \frac{1}{2}g_{ij}([F, P_i]P_j + P_i[F, P_j])$$
$$= \frac{1}{2}g_{ij}P_i[F, P_j] + [F, P_i]\frac{1}{2}g_{ij}P_j$$
$$= \frac{1}{2}(\dot{X}_i\partial_i(F) + \partial_i(F)\dot{X}_i).$$

This completes the proof. □

With this Lemma we see that the quadratic Hamiltonian not only connects the abstract metric coefficients g_{ij} of the non-commutative world with the metric coefficients in classical worlds, but also the basic time derivative formula

$$\dot{F} = \dot{X}_i \partial_i(F)$$

has its correct (symmetrized) non-commutative counterpart. Elsewhere [17] we have said that with the quadratic Hamiltonian, the non-commutative world satisfies the *first constraint.*

One can consider the consequences of the commutator $[X_i, \dot{X}_j] = g_{ij}$, deriving that

$$\ddot{X}_r = G_r + F_{rs}\dot{X}^s + \Gamma_{rst}\dot{X}^s\dot{X}^t,$$

where G_r is the analogue of a scalar field, F_{rs} is the analogue of a gauge field and Γ_{rst} is the Levi-Civita connection associated with g_{ij}. This decompositon of the acceleration is uniquely determined by the given framework. See [16, 14] for the details of this result. Here we discuss the Levi-Civita connection.

A stream of consequences follows by differentiating both sides of the equation

$$g_{ij} = [X_i, \dot{X}_j].$$

We show how the form of the Levi-Civita connection appears naturally.
In the following we shall use D as an abbreviation for d/dt.

The Levi-Civita connection

$$\Gamma_{kij} = (1/2)(\nabla_i g_{jk} + \nabla_j g_{ik} - \nabla_k g_{ij})$$

associated with the g_{ij} comes up almost at once from the differentiation process described above. To see how this happens, view the following calculation where

$$\hat{\partial}_i \hat{\partial}_j F = [X_i, [X_j, F]].$$

We apply the operator $\hat{\partial}_i \hat{\partial}_j$ to the second time derivative of X_k.

Lemma 3.1 Let $\Gamma_{kij} = (1/2)(\nabla_i g_{jk} + \nabla_j g_{ik} - \nabla_k g_{ij})$ where $\nabla_i(F) = [F, \dot{X}_i]$, the covariant derivative generated by $\dot{X}_i = P_i - A_i$. Then

$$\Gamma_{kij} = (1/2)\hat{\partial}_i \hat{\partial}_j \ddot{X}_k.$$

Proof. Note that by the Leibniz rule

$$D([A,B]) = [A,\dot{B}] = [\dot{A},B] + [A,\dot{B}],$$

we have

$$\dot{g_{jk}} = [\dot{X}_j, \dot{X}_k] + [X_j, \ddot{X}_k].$$

Therefore

$$\hat{\partial}_i \hat{\partial}_j \ddot{X}_k = [X_i, [X_j, \ddot{X}_k]]$$

$$= [X_i, \dot{g_{jk}} - [\dot{X}_j, \dot{X}_k]]$$

$$= [X_i, \dot{g_{jk}}] - [X_i, [\dot{X}_j, \dot{X}_k]]$$

(Now use the Jacobi identity $[A,[B,C]] + [C,[A,B]] + [B,[C,A]] = 0.$)

$$= [X_i, \dot{g_{jk}}] + [\dot{X}_k, [X_i, \dot{X}_j]] + [\dot{X}_j, [\dot{X}_k, X_i]]$$

$$= -[\dot{X}_i, g_{jk}] + [\dot{X}_k, [X_i, \dot{X}_j]] + [\dot{X}_j, [\dot{X}_k, X_i]]$$

$$= \nabla_i g_{jk} - \nabla_k g_{ij} + \nabla_j g_{ik}$$

$$= 2\Gamma_{kij}.$$

This completes the proof. □

Remark. The upshot of this derivation is that it confirms our interpretation of

$$g_{ij} = [X_i, \dot{X}_j] = [X_i, P_j] - [X_i, A_j] = \delta_{ij} - \partial A_j / \partial P_i$$

as an abstract form of metric (in the absence of any actual notion of distance in the non-commutative world). *This calls for a re-evaluation and reconstruction of differential geometry based on non-commutativity and the Jacobi identity.* This is differential geometry where the fundamental concept is no longer parallel translation, but rather a non-commutative version of a physical trajectory.

Note that given

$$\Gamma_{kij} = (1/2)(\nabla_i g_{jk} + \nabla_j g_{ik} - \nabla_k g_{ij}),$$

we have

$$\Gamma_{ikj} = (1/2)(\nabla_k g_{ji} + \nabla_j g_{ki} - \nabla_i g_{kj}).$$

Hence

$$\Gamma_{kij} + \Gamma_{ikj} = \nabla_j g_{ik} = \partial_j g_{ik} + [g_{ik}, A_j].$$

In other words, we see that the Levi-Civita connection that we have derived differs from the classical Levi-Civita connection via the replacement of the partial derivatives ∂_j with the covariant derivatives $\nabla_j = \partial_j + A_j$ where $A_j(F) = [F, A_j]$. Since A_j is our analog of a gauge field, this formalism can be compared with the corresponding Levi-Civita connection in Weyl's theory combining aspects of general relativity with electromagnetism, and we see that this is exactly how Weyl did modify the Levi-Civita connection. See [24] Chapter 35, page 290 and [22] p. 85. By following the non-commutative world formalism we have reproduced the core of Weyl's original approach to gauge theory in a new context.

We have already seen how electromagnetism and gauge theories arise in this setting and indeed the motion of a particle as described by the equation [16, 14]

$$\ddot{X}_r = G_r + F_{rs}\dot{X}^s + \Gamma_{rst}\dot{X}^s\dot{X}^t$$

is also the analogue of a particle moving in the Weyl theory where the new Levi-Civita connection controls its acceleration. Note also that our metric coefficients g_{ij} are subject to the curvature associated with the gauge A_i and so undergo holonomy just as did Weyl's metric. It is useful to make these comparisons because the intent of the original Weyl theory was to extend general relativity to include electromagnetism in terms the holonomy associated with the metric. This full extension was not developed due to criticisms of the program and also because the gauge idea related directly to quantum theory in a highly fruitful way where the metric holonomy was replaced by holonomy for the phase of the wave function. Our present formalism suggests a new investigation of these connections that is beyonnd the scope of the present paper.

Nevertheless, note that classical general relativity begins with the standard Levi-Civita connection, and so defining curvature via its interpretation as a parallel displacement. To this end, recall the formalism of parallel translation. The infinitesimal parallel translate of A is denoted by $A' = A + \delta A$ where

$$\delta A^k = -\Gamma^k_{ij} A^i dX^j$$

where here we are writing in the usual language of vectors and differentials with the Einstein summation convention for repeated indices. We assume that the Christoffel symbols satisfy the symmetry condition $\Gamma^k_{ij} = \Gamma^k_{ji}$. The inner product is given by the formula

$$< A, B >= g_{ij} A^i B^j$$

Note that here the bare symbols denote vectors whose coordinates may be indicated by indices. The requirement that this inner product be invariant under parallel displacement is the requirement that $\delta(g_{ij}A^i A^j) = 0$. Calculating, one finds

$$\delta(g_{ij}A^i A^j) = (\partial_k g_{ij})A^i A^j dX^k + g_{ij}\delta(A^i)A^j + g_{ij}A^i\delta(A^j)$$

$$= (\partial_k g_{ij})A^i A^j dX^k - g_{ij}\Gamma^i_{rs}A^r dX^s A^j - g_{ij}A^i \Gamma^j_{rs} A^r dX^s$$

$$= (\partial_k g_{ij})A^i A^j dX^k - g_{ij}\Gamma^i_{rs}A^r A^j dX^s - g_{ij}\Gamma^j_{rs} A^i A^r dX^s$$

$$= (\partial_k g_{ij})A^i A^j dX^k - g_{sj}\Gamma^s_{ik}A^i A^j dX^k - g_{is}\Gamma^s_{jk} A^i A^j dX^k$$

Hence

$$(\partial_k g_{ij}) = g_{sj}\Gamma^s_{ik} + g_{is}\Gamma^s_{jk}.$$

From this it follows that

$$\Gamma_{ijk} = g_{is}\Gamma^s_{jk} = (1/2)(\partial_k g_{ij} - \partial_i g_{jk} + \partial_j g_{ik}).$$

Certainly these notions of variation can be imported into our abstract context. The question remains how to interpret the new connection that arises when we replace ∂_i with $\nabla_i = \partial_i + A_i$. This yields our new Levi-Civita connection (here denoted by the old symbol).

$$\Gamma_{ijk} = g_{is}\Gamma^s_{jk} = (1/2)(\nabla_k g_{ij} - \nabla_i g_{jk} + \nabla_j g_{ik}).$$

One could forge forward and define a new covariant derivative by the formula

$$\hat{\nabla}_i X^j = \partial_i X^j + \Gamma^j_{ki} X^k$$

and then attempt to head toward a generalization of general relativity via its curvature tensors. The question is how the curvature of this connection interfaces with the gauge potentials that gave rise to the metric in the first place. The theme of this investigation has the flavor of gravity theories with a qauge theoretic background. The difficulty is that due to the nature of the new covariant derivatives there is not known (to this author) a suitable analogue for the Einstein tensor. The new Riemann tensor will no longer have the many symmetries of the classical Riemann tensor and a simple copy of the original Einstein tensor will not have vanishing divergence.

All this said, the above becomes a significant program for further investigation. It would appear that this program was implicit in Weyl's original work as well, and examination of his papers and the book "Space Time Matter" [24] suggests that he did not carry out a full unification of his gauge theory with general relativity. Our approach suggests a new start on this problem.

4 Recapitulation - Curvature, Jacobi Identity and the Levi-Civita Connection

In this section, we go back to basics and examine the context of calculus defined via commutators. We recapitulate what we have accomplished so far, and set the stage for the next level of structure. We shall use a partially index-free notation. In this notation, we avoid nested subscripts by using different variable names and then using these names as subscripts to refer to the relevant variables. Thus we write X and Y instead of X_i and X_j, and we write g_{XY} instead of g_{ij}. It is assumed that the derivation DX has the form $DX = [X, J]$ for some J.

The bracket $[A, B]$ is not assumed to be a commutator. It is assumed to satisfy the Jacobi identity, bilinearity in each variable, and the Leibniz rule for all functions of the form $\delta_K(A) = [A, K]$. That is we assume that

$$\delta_K(AB) = \delta_K(A)B + A\delta_K(B).$$

Recall that in classical differential geometry one has the notion of a covariant dervative, defined by taking a difference quotient using parallel translation via a connection. Covariant derivatives in different directions do not necessarily commute. The commutator of covariant derivatives gives rise to the curvature tensor in the form

$$[\nabla_i, \nabla_j]X^k = R^k_{lij}X^l.$$

If derivatives do not commute then we regard their commutator as expressing a curvature. In our non-commutative context this means that curvature arises *prior* to any notion of covariant derivatives since *even the basic derivatives do not commute.*

We shall consider derivatives in the form

$$\nabla_X(A) = [A, \Lambda_X].$$

Examine the following computation:

$$\nabla_X \nabla_Y F = [[F, \Lambda_Y], \Lambda_X] = -[[\Lambda_X, F], \Lambda_Y] - [[\Lambda_Y, \Lambda_X], F]$$

$$= [[F, \Lambda_X], \Lambda_Y] + [[\Lambda_X, \Lambda_Y], F]$$
$$= \nabla_Y \nabla_X F + [[\Lambda_X, \Lambda_Y], F].$$

Thus

$$[\nabla_X, \nabla_Y]F = R_{XY}F$$

where

$$R_{XY}F = [[\Lambda_X, \Lambda_Y], F].$$

We can regard R_{XY} as a curvature operator.

The analog in this context of flat space is abstract quantum mechanics! That is, we assume position variables (operators) X, Y, $\cdots$ and momentum variables (operators) P_X, P_Y, $\cdots$ satisfying the equations below.

$$[X, Y] = 0$$

$$[P_X, P_Y] = 0$$

$$[X, P_Y] = \delta_{XY}$$

where δ_{XY} is equal to one if X equals Y and is zero otherwise. We define

$$\partial_X F = [F, P_X]$$

and

$$\partial_{P_X} F = [X, F].$$

In the context of the above commutation relations, note that these derivatives behave correctly in that

$$\partial_X(Y) = \delta_{XY}$$

and

$$\partial_{P_X}(P_Y) = \delta_{XY}$$

$$\partial_{P_X}(Y) = 0 = \partial_X(P_Y)$$

with the last equations valid even if $X = Y$. Note also that iterated partial derivatives such as $\partial_X \partial_Y$ commute. Hence the curvature R_{XY} is equal to zero. We shall regard these position and momentum operators and the corresponding partial derivatives as an abstract algebraic substitute for flat space.

With this reference point of (algebraic, quantum) flat space we can define

$$\hat{P}_X = P_X - A_X$$

for an arbitrary algebra-valued function of the variable X. In indices this would read

$$\hat{P}_i = P_i - A_i,$$

and with respect to this deformed momentum we have the covariant derivative

$$\nabla_X F = [F, \hat{P}_Y] = [F, P_Y + A_Y] = \partial_Y F + [F, A_Y].$$

The curvature for this covariant derivative is given by the formula

$$R_{XY}F = [\nabla_X, \nabla_Y]F = [[\lambda_X, \lambda_Y], F]$$

where $\lambda_X = P_X - A_X$. Hence

$$R_{XY} = [P_X - A_X, P_Y - A_Y] = -[P_X, A_Y] - [A_X, P_Y] + [A_X, A_Y]$$

$$= \partial_X A_Y - \partial_Y A_X + [A_X, A_Y].$$

With indices this reads

$$R_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j].$$

and the reader will note that this has the abstract form of the curvature of a Yang-Mills gauge field, and specifically the form of the electromagnetic field when the potentials A_i and A_j commute with one another.

Continuing with this example, we compute

$$[X, \hat{P_Y}] = [X, P_Y - A_Y] = \delta_{XY} - [X, A_Y].$$

Let

$$g_{XY} = \delta_{XY} - [X, A_Y]$$

so that

$$[X, \hat{P_Y}] = g_{XY}.$$

We will shortly consider the form of this general case, but first it is useful to restrict to the case where $[X, A_Y] = 0$ so that $g_{XY} = \delta_{XY}$ (for the space coordinates). In order to enter this domain, we set

$$\dot{X} = DX = \hat{P_X} = P_X - A_X.$$

We then have

$$[X_i, X_j] = 0$$

$$[X_i, \dot{X_j}] = \delta_{ij}$$

and

$$R_{ij} = [\dot{X}_i, \dot{X}_j] = \partial_i A_j - \partial_j A_i + [A_i, A_j].$$

Generalizing, we wish to examine the structure of the following special axioms for a bracket.

$$[X, DY] = g_{XY}$$
$$[X, Y] = 0$$
$$[Z, g_{XY}] = 0$$
$$[g_{XY}, g_{ZW}] = 0$$

Note that

$$Dg_{YZ} = D[Y, DZ] = [DY, DZ] + [Y, D^2 Z].$$

and that $D[X, g_{XY}] = 0$ implies that

$$[g_{XY}, DZ] = [Z, Dg_{XY}].$$

Define two types of derivations as follows

$$\nabla_X(F) = [F, DX]$$

and

$$\nabla_{DX}(F) = [X, F].$$

These are dual with respect to g_{XY} and will act like partials with respect to these variables in the special case when g_{XY} is a Kronecker delta, δ_{XY}. If the form g_{XY} is invertible, then we can rewrite these derivations by contracting the inverse of g to obtain standard formal partials.

$$\nabla_{DX}\nabla_{DY} D^2 Z = [X, [Y, D^2 Z]]$$
$$= [X, Dg_{YZ} - [DY, DZ]] = [X, Dg_{YZ}] - [X, [DY, DZ]]$$
$$= [g_{YZ}, DX] - [X, [DY, DZ]]$$
$$= \nabla_X(g_{YZ}) - [X, [DY, DZ]].$$

Now use the Jacobi identity on the second term and obtain

$$\nabla_{DX}\nabla_{DY} D^2 Z = \nabla_X(g_{YZ}) + [DZ, [X, DY]] + [DY, [DZ, X]]$$
$$= \nabla_X(g_{YZ}) - \nabla_Z(g_{XY}) + \nabla_Y(g_{XZ}).$$

This is the formal Levi-Civita connection.

At this stage we face once again the mystery of the appearance of the Levi-Civita connection. We have seen in this section that it is quite natural for curvature in the form of the non-commutativity of derivations to appear at the outset in a non-commutative formalism. We have also see that this curvature and connection can be understood as a measurement of the deviation of the theory from the flat commutation relations of ordinary quantum mechanics. Electromagnetism and Yang-Mills theory can be seen as the theory of the curvature introduced by such a deviation. On the other hand, from the point of view of metric differential geometry, the Levi-Civita connection is the unique connection that preserves the inner product defined by the metric under the parallel translation defined by the connection.

5 Appendix – Einstein's Equations and the Bianchi Identity

The purpose of this section is to show how the Bianchi identity (see below for its definition) appears in the context of non-commutative worlds. The Bianchi identity is a crucial mathematical ingredient in general relativity. We shall begin with a quick review of the mathematical structure of general relativity (see for example [4]) and then turn to the context of non-commutative worlds.

The basic tensor in Einstein's theory of general relativity is

$$G^{ab} = R^{ab} - \frac{1}{2}Rg^{ab}$$

where R^{ab} is the Ricci tensor and R the scalar curvature. The Ricci tensor and the scalar curvature are both obtained by contraction from the Riemann curvature tensor R^a_{bcd} with $R_{ab} = R^c_{abc}, R^{ab} = g^{ai}g^{bj}R_{ij}$, and $R = g^{ij}R_{ij}$. Because the Einstein tensor G^{ab} has vanishing divergence, it is a prime candidate to be proportional to the energy momentum tensor $T^{\mu\nu}$. The Einstein field equations are

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} = \kappa T^{\mu\nu}.$$

The reader may wish to recall that the Riemann tensor is obtained from the commutator of a covariant derivative ∇_k, associated with the Levi-Civita connection $\Gamma^i_{jk} = (\Gamma_k)^i_j$ (built from the space-time metric g_{ij}). One has

$$\lambda_{a:b} = \nabla_b \lambda_a = \partial_b \lambda_a - \Gamma^d_{ab}\lambda_d$$

or

$$\lambda_{:b} = \nabla_b \lambda = \partial_b \lambda - \Gamma_b \lambda$$

for a vector field λ. With

$$R_{ij} = [\nabla_i, \nabla_j] = \partial_j \Gamma_i - \partial_i \Gamma_j + [\Gamma_i, \Gamma_j],$$

one has

$$R^a_{bcd} = (R_{cd})^a_b.$$

(Here R_{cd} is *not* the Ricci tensor. It is the Riemann tensor with two internal indices hidden from sight.)

One has explicitly that [4]

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}(g_{\mu\sigma,\nu\rho} - g_{\nu\sigma,\mu\rho} - g_{\mu\rho,\nu\sigma} + g_{\nu\rho,\mu\sigma}) + \Gamma_{\beta\mu\sigma}\Gamma^{\beta}_{\nu\rho} + \Gamma_{\beta\mu\rho}\Gamma^{\beta}_{\nu\rho}.$$

Many symmetries of the Riemann tensor follow from this formula. If the derivatives in this formula are replaced by covariant derivatives, the symmetries do not all survive and the story we are about to tell about the divergence of the Einstein tensor will have to be modified. That is a project for future work.

One way to understand the mathematical source of the Einstein tensor, and the vanishing of its divergence, is to see it as a contraction of the Bianchi identity for the Riemann tensor. The Bianchi identity states

$$R^a_{bcd:e} + R^a_{bde:c} + R^a_{bec:d} = 0$$

where the index after the colon indicates the covariant derivative. Note also that this can be written in the form

$$(R_{cd:e})^a_b + (R_{de:c})^a_b + (R_{ec:d})^a_b = 0.$$

The Bianchi identity is a consequence of local properties of the Levi-Civita connection and consequent symmetries of the Riemann tensor. One relevant symmetry of the Riemann tensor is the equation $R^a_{bcd} = -R^a_{bdc}$.

We will not give a classical derivation of the Bianchi identity here, but it is instructive to see how its contraction leads to the Einstein tensor. To this end, note that we can contract the Bianchi identity to

$$R^a_{bca:e} + R^a_{bae:c} + R^a_{bec:a} = 0$$

which, in the light of the above definition of the Ricci tensor and the symmetries of the Riemann tensor is the same as

$$R_{bc:e} - R_{be:c} + R^a_{bec:a} = 0.$$

Contract this tensor equation once more to obtain

$$R_{bc:b} - R_{bb:c} + R^a_{bbc:a} = 0,$$

and raise indices

$$R^b_{c:b} - R_{:c} + R^{ab}_{bc:a} = 0.$$

Further symmetry gives

$$R^{ab}_{bc:a} = R^{ba}_{cb:a} = R^a_{c:a} = R^b_{c:b}.$$

Hence we have

$$2R^b_{c:b} - R_{:c} = 0,$$

which is equivalent to the equation

$$(R^b_c - \frac{1}{2}R\delta^b_c)_{:b} = G^b_{c:b} = 0.$$

From this we conclude that $G^{bc}_{:b} = 0$. The Einstein tensor has appeared on the stage with vanishing divergence, courtesy of the Bianchi identity!

Bianchi Identity and Jacobi Identity. Now lets turn to the context of non-commutative worlds. We have infinitely many possible convariant derivatives, all of the form

$$F_{:a} = \nabla_a F = [F, N_a]$$

for some N_a elements in the non-commutative world. Choose any such covariant derivative. Then, as in the introduction to this paper, we have the curvature

$$R_{ij} = [N_i, N_j]$$

that represents the commutator of the covariant derivative with itself in the sense that $[\nabla_i, \nabla_j]F = [[N_i, N_j], F]$. Note that R_{ij} is not a Ricci tensor, but rather the indication of the external structure of the curvature without any particular choice of linear representation (as is given in the classical case as described above). We then have the Jacobi identity

$$[[N_a, N_b], N_c] + [[N_c, N_a], N_b] + [[N_b, N_c], N_a] = 0.$$

Writing the Jacobi identity in terms of curvature and covariant differention we have

$$R_{ab:c} + R_{ca:b} + R_{bc:a}.$$

Thus in a non-commutative world, every covariant derivative satisfies its own Bianchi identity. This gives an impetus to study general relativity in non-commutative worlds by looking for covariant derivatives that satisfy the symmetries of the Riemann tensor and link with a metric in an appropriate way. We have only begun this aspect of the investigation. The point of this section has been to show the intimate relationship between the Bianchi idenity and the Jacobi identity that is revealed in the context of non-commutative worlds.

6 Appendix - Discrete Calculus Reformulated with Commutators

There are many motivations for replacing derivatives by commutators. If $f(x)$ denotes (say) a function of a real variable x, and $\tilde{f}(x) = f(x+h)$ for a fixed increment h, define the *discrete derivative* Df by the formula $Df = (\tilde{f} - f)/h$, and find that the Leibniz rule is not satisfied. One has the basic formula for the discrete derivative of a product:

$$D(fg) = D(f)g + \tilde{f}D(g).$$

Correct this deviation from the Leibniz rule by introducing a new non-commutative operator J with the property that

$$fJ = J\tilde{f}.$$

Define a new discrete derivative in an extended non-commutative algebra by the formula

$$\nabla(f) = JD(f).$$

It follows at once that

$$\nabla(fg) = JD(f)g + J\tilde{f}D(g) = JD(f)g + fJD(g) = \nabla(f)g + f\nabla(g).$$

Note that

$$\nabla(f) = (J\tilde{f} - Jf)/h = (fJ - Jf)/h = [f, J/h].$$

In the extended algebra, discrete derivatives are represented by commutators, and satisfy the Leibniz rule. One can regard discrete calculus as a subset of non-commutative calculus based on commutators.

Discrete Measurement. Consider a time series $\{X, X', X'', \cdots\}$ with commuting scalar values. Let

$$\dot{X} = \nabla X = JDX = J(X' - X)/\tau$$

where τ is an elementary time step (If X denotes a times series value at time t, then X' denotes the value of the series at time $t + \tau$.). The shift operator J is defined by the equation $XJ = JX'$ where this refers to any point in the time series so that $X^{(n)}J = JX^{(n+1)}$ for any non-negative integer n. Moving J across a variable from left to right, corresponds to one tick of the clock. This discrete, non-commutative time derivative satisfies the Leibniz rule.

This derivative ∇ also fits a significant pattern of discrete observation. Consider the act of observing X at a given time and the act of observing (or obtaining) DX at a given time. Since X and X' are ingredients in computing $(X' - X)/\tau$, the numerical value associated with DX, it is necessary to let the clock tick once, Thus, if one first observe X and then obtains DX, the result is different (for the X measurement) if one first obtains DX, and then observes X. In the second case, one finds the value X' instead of the value X, due to the tick of the clock.

1. Let $\dot{X}X$ denote the sequence: observe X, then obtain $\dot{X}$.
2. Let $X\dot{X}$ denote the sequence: obtain $\dot{X}$, then observe X.

The commutator $[X, \dot{X}]$ expresses the difference between these two orders of discrete measurement. In the simplest case, where the elements of the time series are commuting scalars, one has

$$[X, \dot{X}] = X\dot{X} - \dot{X}X = J(X' - X)^2/\tau.$$

Thus one can interpret the equation

$$[X, \dot{X}] = Jk$$

(k a constant scalar) as

$$(X' - X)^2/\tau = k.$$

This means that the process is a walk with spatial step

$$\Delta = \pm\sqrt{k\tau}$$

where k is a constant. In other words, one has the equation

$$k = \Delta^2/\tau.$$

This is the diffusion constant for a Brownian walk. A walk with spatial step size Δ and time step τ will satisfy the commutator equation above exactly when the square of the spatial step divided by the time step remains constant. This shows that the diffusion constant of a Brownian process is a structural property of that process, independent of considerations of probability and continuum limits.

7 Appendix - On Weyl's Line Element for Electromagnetism

In this appendix we review the essentials of Weyl's approach to electromagnetism. This is lucidly explained in [24, 22, 5]. Consider a line element for spacetime of the form

$$\lambda = Fdx + Gdy + Hdz - \phi dt$$

Regard λ as a differential 1-form. Then (with wedge products of differentials so that $dxdy = -dydx$ and so on) we have

$$d\lambda = (G_x - F_y)dxdy + (H_x - F_y)dxdz + (H_y - G_z)dydz$$

$$-(F_t + \phi_x)dxdt - (G_t + \phi_y)dydt - (H_t + \phi_z)dzdt.$$

Hence, if we set

$$d\lambda = B_1 dydz - B_2 dxdz + B_3 dxdy + E_1 dxdt + E_2 dydt + E_3 dzdt$$

and $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$, $A = (A_1, A_2, A_3) = (F, G, H)$ then

$$E = -\nabla\phi - \partial A/\partial t,$$

$$B = \nabla \times A.$$

Thus the differential of the line element produces electric and magnetic fields with the space parts of the line element acting as the vector potential and the time part acting as the scalar potential. Furthermore, one finds that $d^2\lambda = 0$ (consequence of the properties of differential forms) and with $d\lambda$ in terms of E and B the equation $d^2\lambda = 0$ becomes

$$\nabla \bullet B = 0,$$

$$\nabla \times E + \partial B/\partial t = 0.$$

Thus indeed the line element does represent the potentials for electromagnetism, and the equation $d^2\lambda = 0$ produces Maxwell's equations. The other two Maxwell equations

$$\nabla \bullet E = \rho,$$

$$\nabla B - \partial E/\partial t = J$$

can be regarded as the definitions of the charge density ρ and the current J.

This means that we can regard a spacetime line element λ as the holder of the structure that gives rise to the electromagnetic field. If $d\lambda = 0$ then the line element will have no holonomy, no change along different paths from one point to another. But if the E and B fields defined by $d\lambda$ are non-zero, then distances will vary depending upon the path taken between two points.

Thus the curvature of this gauge field was identified by Weyl as the electromagnetic field, and he worked on a formalism to unify it with general relativity. The inutitive idea was that in moving from one point of spacetime to another there was spacetime curvature as in general relativity and also curvature that connoted the electromagnetic field via the varation of the line element. Eventually all these considerations were integrated into physics in a different way by regarding that line element as representing the phase of the quantum wave function. Unifications of gauge theory and general relativity have proceeded in different directions. Here we have begun a different way to formulate the Weyl idea in terms of non-commutative worlds, and it remains to see the full consequences of our approach.

We remark that the standard generalization of the differential 1-form λ is to write $A = \sum_i A_i dx^i$ as a gauge connection where the A_i do not commute with one another and take the form $A_i(x) = \sum_a A_i^a(x) T_a$ where the T_a run over a basis for a matrix representation of the Lie algebra of the gauge group, and the $A_i^a(x)$ are smooth functions on the spacetime manifold. Then the curvature of the gauge connection is $F = dA + A \wedge A$ where $\wedge$ denotes the wedge product of differential forms. This generalizes the way we have just described the electromagnetic field in terms of the Weyl differential 1-form and gives rise to the Yang-Mills fields. At the level of non-commutative worlds we can consider abstract differential forms $A = \sum_i A_i dx^i$ without assuming that the A_i are represented in terms of a specific classical gauge group. By the same token, we can examine the structure of covariant derivatives of the form $\nabla_i = \partial_i + A_i$ and indeed one finds directly that

$$[\nabla_i, \nabla_j] = \partial_i A_j - \partial_j A_i + [A_i, A_j].$$

In this way, the formalism of the differential forms and the formalism of the commutators of covariant derivatives come together.

Weyl's interpretation of the properties of the line element $A = \lambda$ was that an integral along a path from event p to event q, $\int_p^q A$, would be path dependent and that this would represent changes in spacetime distance between points depending on the path (history) between them. This path dependence would be a manifestation of the electromagnetic field dA (In Weyl's form $A \wedge A = 0$). Einstein criticized the theory on these grounds and a new interpretation eventually appeared. The new interpretation can be summarized by multiplying the integral by the square root of negative unity, $i \int_p^q A$, and interpreting it via $e^{i \int_p^q A}$ as a change of phase of a quantum wave function associated with the gauge connection. In this form, many years later [1] exactly this effect was discovered for electromagnetism, and it became known as the Aharanov-Bohm effect. The interpretation of the gauge connection for phases of quantum wave functions became an established part of physics, vindicating Weyl's intuitions, albeit with a shift of interpretation [25]. It is only more recently that gravity is seen in relation to gauge fields. The work of Abhay Ashtekar, Carol Rovelli and Lee Smolin has led to the emergence of the field of Loop Quantum Gravity [2, 20, 23] where a gauge formulation of quantum gravity has non-trivial holonomies for macroscopic loops that are central to the theory. It should be mentioned that in the work of Witten [26] these kinds of holonomies are closely related to topological invariants of knots, links and three-manifolds. See also [7].

A particularly interesting theory to examine in our non-commutative context is the loop quantum gravity version of general relativity that uses the Ashtekar variables [2, 20, 23]. In that theory the metric is expressed in terms of a gauge group and the gauge holonomy plays a significant role in the physics and its relation to topology. We intend to examine this structure in a sequel to the present paper.

References

[1] Aharonov, Y; Bohm, D [1959]. "Significance of electromagnetic potentials in quantum theory". Physical Review. 115 (3): 485491.

[2] Baez J. and Muniain J. P. [1994], "Gauge Fields, Knots and Gravity", World Scientific, Singapore.

[3] Dyson, F. J. [1990], Feynman's proof of the Maxwell Equations, *Am. J. Phys.* 58 (3), March 1990, 209-211.

[4] Dirac, P.A.M. [1975], General Theory of Relativity, John Wiley and Sons, New York.

[5] Eddington, Sir Arthur, [1953] "Space, Time and Gravitation", Harper and Row Pub., New York.

[6] Hughes, R. J. [1992], On Feynman's proof of the Maxwell Equations, *Am. J. Phys.* 60, (4), April 1992, 301-306.

[7] Kauffman,L. H.[1991,1994], *Knots and Physics,* World Scientific Pub.

[8] Kauffman,L. H. and Noyes,H. Pierre [1996], Discrete Physics and the Derivation of Electromagnetism from the formalism of Quantum Mechanics, *Proc. of the Royal Soc. Lond. A*, **452**, pp. 81-95.

[9] Kauffman,L. H. and Noyes,H. Pierre [1996], Discrete Physics and the Dirac Equation, *Physics Letters A*, 218 ,pp. 139-146.

[10] Kauffman, L. H.[1996], Quantum electrodynamic birdtracks, *Twistor Newsletter Number 41*

[11] Kauffman, L. H. [1998], Noncommutativity and discrete physics, *Physica D* 120 (1998), 125-138.

[12] Kauffman, L. H. [1998], Space and time in discrete physics, *Intl. J. Gen. Syst.* Vol. 27, Nos. 1-3, 241-273.

[13] Kauffman, L. H. [1999], A non-commutative approach to discrete physics, in *Aspects II - Proceedings of ANPA 20*, 215-238.

[14] Kauffman, L. H. [2000], Differential geometry in non-commutative worlds, in "Quantum Gravity", ed. by Fauser, Tolksdodrf, and Zeidler, 2000, pp. 61-76.

[15] Kauffman, L. H. [2003], Non-commutative calculus and discrete physics, in *Boundaries- Scientific Aspects of ANPA 24*, 73-128.

[16] Kauffman, L. H. [2004], Non-commutative worlds, *New Journal of Physics 6*, 2-46.

[17] Kauffman, Louis H. [2018], Non-commutative worlds and classical constraints. Entropy 20 (2018), no. 7, Paper No. 483, 25 pp.

[18] A. M. Deakin and L. H. Kauffman [2018], Cosmological theories of the extra terms, in Chiasmus - Proceedings of ANPA No. 37 and 38, May 2018. pp. 107-202.

[19] Montesinos, M. and Perez-Lorenzana, A., [1999], Minimal coupling and Feynman's proof, arXiv:quant-phy/9810088 v2 17 Sep 1999.

[20] Rovelli, C. [2004] "Quantum Gravity", Cambridge University Press, Cambridge, UK (2004).

[21] Tanimura,Shogo [1992], Relativistic generalization and extension to the non-Abelian gauge theory of Feynman's proof of the Maxwell equations, *Annals of Physics, vol. 220*, pp. 229-247.

[22] O'Raifeartaigh, L. [1997], "The Dawning of Gauge Theory", Princeton University Press, Princeton, N. J.

[23] Gambini, R. and Pullin, J. [1996], "Loops, knots gauge theories and quantum gravity", Cambridge University Press, Cambridge , UK (1996).

[24] Weyl, H. [1952], Space - Time - Matter, Dover Publications (1952), New York.

[25] Yang, C. N. [1986] Hermann Weyl's contributions to physics, in "Hermann Weyl 1885-1955", edited by Chandrasekharan, K., Springer-Verlag, London, Paris, Tokyo (1986).

[26] Witten, E. [1989], Quantum field theory and the Jones polynomial, Commun. Math. Phys. 121, 351-399 (1989).