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Precausal quantum mechanics

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It is pointed out that both classical Wheeler-Feynman electrodynamics and its finite quantized generalization inevitably lead to microscopic causality violation. As there is some evidence for such effects in proton Compton scattering, there is possibly reason to prefer such absorber theories of action at a distance over field theories as the more reasonable microscopic description of nature.

INTRODUCTION

In a pair of remarkable articles,[1,2] Wheeler and Feynman demonstrated how the Schwarzschild-Tetrode-Fokker action at a distance theory could appear almost completely identical to the usual Maxwell-Lorentz-Dirac[3] field theory of classical electrodynamics. Such theories were later extended to the quantum regime by Hoyle and Narlikar[4-6] and Davies.[7-9] A necessary condition for the viability of such action-at-a-distance theories is that every photon ever emitted must eventually be absorbed. Such theories are thus known as absorber theories of radiation. Whether this condition is true or not is a cosmological property[10] and cannot yet be conclusively decided.[11]

Although having great similarities, there are some very significant differences between action at a distance and field theories. One of the most important differences is that the action-at-a-distance quantities which obey the field equations do not have their own independent degrees of freedom. Loosely speaking, these "adjunct fields" contain a summary of the remote influence of all the other particles in the universe at a given space-time point. Quantum field theories, in contrast, invest every point in space with independent degrees of freedom (often represented as those of a harmonic oscillator). One of the most basic difficulties of quantum field theory is precisely the fact that the zero-point energy of these degrees of freedom at every point in space leads to an infinite vacuum energy. The infinite vacuum energy remains today as one of the most intractable problems of quantum field theory. In contrast, an action-at-a-distance formulation instantly eliminates the vacuum-energy problem, as the adjunct fields do not have independent degrees of freedom to be quantized. Furthermore, several effects which are conventionally thought to require the existence of field degrees of freedom, such as spontaneous emission and the dynamic stark effect,[12] have been shown to be describable completely within the framework of action at a distance. Absorber theory is also a very serviceable framework for discussions of the standard paradoxes of quantum mechanics as well,[13-16] and a review of the connection between absorber theory and "time's arrow," i.e., the distinction between past and future, has recently been given.[17] A review of absorber theory, with numerous further references, has been given by Pegg.[18]

There is a feature of the classical absorber theory of Wheeler and Feynman which has not been retained in the existing quantized forms of absorber theory. The original absorber theory does not involve any intrinsic self-action. The classical force of radiative reaction, which appears to be a self-action, is found to be only an indirect result of the interaction with all the other remote particles in the environment. Because the classical absorber theory involves no intrinsic self-action, there are two direct byproducts, viz., the theory is completely finite, as the would-be self-fields arbitrarily close to a given particle are not present, and there is an inevitable violation of microscopic causality, in that the force of radiative reaction contains both advanced and retarded fields.

Barring experimental evidence for microscopic causality violation, it is no wonder that the quantum versions of absorber theory retained intrinsic self-action, and thus causality. To quote Pegg,[18] "The absence of self-action of the electron, one of the major features of the original theory, no longer appears necessary or even useful. The original hope that the omission of self-action could solve some of the divergence problems of quantum electrodynamics has not been realized."

By retaining an intrinsic self-interaction in the quantized absorber theories, these theories retain both the causality properties and the divergences of the corresponding quantum field theory. Without such an intrinsic self-interaction, these theories inevitably involve microscopic causality violation, as we will discuss below. As we have recently observed that there is substantial evidence in favor of microscopic causality violation,[19,20] we are led to suggest that the absorber theories without intrinsic self-interaction are strongly indicated as being a more natural framework for the description of nature. Finally, if the data should turn out to be merely fortuitously indicating microscopic causality violation, and should subsequent measurements establish that there is no such violation, then this would be substantial evidence against such absorber theories.

In this article we discuss a modified Lagrangian formulation of classical mechanics which is able to accomo-

date dissipative forces, including radiative reaction, in a natural way. We then quantize this Lagrangian theory by the method of path integrals, with particular attention focused on the causality-violating, "precausal" features of the problem. Our approach contrasts with the work of Hoyle and Narlikar, first in that it is considerably simpler, but more importantly in that we explicitly retain the causality-violating features of the classical theory. We end the present article by establishing some connections between our precausal quantum mechanics and conventional quantum mechanics. For simplicity, in the present article we omit all consideration of relativistic effects. The generalization to relativistic quantum mechanics turns out to require such substantial modifications in formulation that we leave all discussion of such effects to a following article.[21]

PRECAUSAL CLASSICAL MECHANICS

As first shown by Dirac,[3] a completely finite classical radiation theory can be derived from the Fokker action

$$S=-\sum_a \int m_a da+\sum_{\substack{a,b\\a<b}} \int\int e_a e_b \delta(s_{ab}^2) da_\mu db_\mu \ , \tag{1}$$

by a principle of stationary action: $\delta S=0$, and the imposition of a boundary condition that the sum of the so-called "in" and "out" fields vanish. Dirac's boundary condition was introduced based on an assumption of a fundamental underlying time symmetry, but remained somewhat mysterious until the work of Wheeler and Feynman. In Eq. (1) and hereafter, we use Einstein summation convention and a system of units for which $h/2\pi$ and c are both unity. The four-vector displacement squared between the points a and b is $s_{ab}^2=(a_\mu-b_\mu)(a_\mu-b_\mu)$, and the differential world line element $da=(-da_\mu da_\mu)^{1/2}=(dt^2-d\mathbf{x}^2)^{1/2}$. In terms of the *definitions*

$$A_\nu^{(b)}(x)=e_b \int \delta(s_{bx}^2) db_\nu \tag{2}$$

and

$$F_{\nu\mu}^{(b)}(x)=\frac{\partial A_\mu^{(b)}(x)}{\partial x_\nu}-\frac{\partial A_\nu^{(b)}(x)}{\partial x_\mu} \ , \tag{3}$$

the four-vector equation of motion for particle a is, with dots indicating derivatives with respect to proper time,

$$m_a \ddot{a}_\nu=e_a \sum_{(b\neq a)} F_{\nu\mu}^{(b)}(a)\dot{a}_\mu \ . \tag{4}$$

The Maxwell equations for the quantities $F_{\mu\nu}$,

$$\frac{\partial F_{\mu\nu}^{(b)}(x)}{\partial x_\nu}=4\pi J_\mu^{(b)}(x) \ , \tag{5}$$

follow tautologically from the definitions, with the currents defined by

$$J_\mu^{(b)}(x)=e_b \int_{-\infty}^{\infty} \delta(x_1-b_1)\delta(x_2-b_2)\delta(x_3-b_3) \times\delta(x_4-b_4)\dot{b}_\mu(\beta)d\beta \ . \tag{6}$$

Although the particle equations of motion (4) and the field equations (5) appear identical to those derived from the Maxwell theory, there are several very important differences. First, we do not have a gauge theory and the "fields" do not contain any independent degrees of freedom of their own, as they are entirely defined in terms of the world lines of the particles. Second, the fields are equal superpositions of advanced and retarded fields. Third, there is no self-action, and therefore no singularity in the field at the loci of the particles.

It was shown by Wheeler and Feynman[1,2] how the time-symmetric fields entering in (4) may lead to apparently fully retarded interactions. The requirement that any light emitted by a system of particles eventually be absorbed by that system of particles leads to the result that the difference of retarded and advanced fields must vanish everywhere, i.e.,

$$\sum_b F_{\text{ret}}^{(b)}-F_{\text{adv}}^{(b)}=0 \ . \tag{7}$$

This response condition may either be merely assumed *a priori* or, by assuming any number of suitable cosmological models, it may be shown to follow from the properties of the universe.[6,10,11] Any model universe with the property (7) is said to be a "perfect absorber."

Given the condition (7), the fields which enter the equations of motion (4) can be rewritten

$$\begin{aligned}\sum_{(b\neq a)} &(F_{\text{ret}}^{(b)}+F_{\text{adv}}^{(b)})/2 \\ &=\sum_{(b\neq a)}(F_{\text{ret}}^{(b)}+F_{\text{adv}}^{(b)})/2+\sum_{(b\neq a)}(F_{\text{ret}}^{(b)}-F_{\text{adv}}^{(b)})/2 \\ &\quad+(F_{\text{ret}}^{(a)}-F_{\text{adv}}^{(a)})/2 \\ &=\sum_{(b\neq a)} F_{\text{ret}}^{(b)}+(F_{\text{ret}}^{(a)}-F_{\text{adv}}^{(a)})/2 \ .\end{aligned} \tag{8}$$

Thus the fully time-symmetric set of fields of all particles besides a is equivalent to the sum of the retarded fields of all other particles plus an apparent self-action term.

An essential point is that there is a certain unavoidable degree of microscopic causality violation introduced by absorber theory, and it is intimately connected with the absence of intrinsic self-interaction. Although the net field experienced by particle a consists of a sum of purely retarded fields from all the remote b particles, there still remains the advanced field of particle a itself in the radiative reaction term. It is well known that in the classical electromagnetic theory of Dirac,[3] the preacceleration of the electron originates precisely from this very term. This point was noted by Wheeler and Feynman in the penultimate sentence of their first paper,[1] which we quote: "In such a system the phenomenon of pre-acceleration appears as the sole evidence of the advanced effects of the theory of action at a distance." If the sum in (1) extended over *all* particles, i.e., it included intrinsic self-action, then the series of equations leading to retarded interactions plus radiative reaction seen in (8) would instead lead to fully retarded interactions everywhere, and no radiative reaction. Such a theory would be completely causal, and due to the infinite self-interaction, would contain divergences. The fact that it

contains divergences is obvious from the vanishing of the argument of the δ function in the hypothetical term with $a=b$ in expression (1).

The quantum versions of the Wheeler-Feynman theory can also be cast in the same general form in terms of advanced and retarded interactions as the classical theory.[5,6] By including intrinsic self-action, these treatments of quantized absorber theory have produced theories exactly equivalent to the conventional quantum field theories including the feature of being completely causal, and containing divergences. Should they instead not have intrinsic self-interaction, we expect that the quantum theories too will contain, in general, microscopic causality violation resulting from the corresponding quantum generalizations of the radiative reaction terms. In the remainder of this paper we consider the nonrelativistic quantization of Wheeler-Feynman electrodynamics, with particular attention paid to the radiative reaction terms responsible for microscopic causality violation which results from the absence of intrinsic self-interaction.

As shown by Dirac, the self-action term (8) can be evaluated as

$$(F_{\text{ret}}^{(a)}-F_{\text{adv}}^{(a)})/2=\tfrac{2}{3}e_a(a_\nu\ddot{a}_\mu-\ddot{a}_\nu a_\mu)\ , \tag{9}$$

so that the equation of motion for particle a becomes

$$m_a\ddot{a}_\nu=e_a\sum_{(b\neq a)}[F_{\nu\mu}^{(b)}(a)]_{\text{ret}}\dot{a}_\mu+2e_a^2/3(\dddot{a}_\nu+\dot{a}_\nu\ddot{a}_\mu\dot{a}_\mu)\ . \tag{10}$$

In this differential equation form, it is the third-order time derivatives which lead to preacceleration phenomena in the classical theory. The radiative reaction force in (10) can be viewed in either of two complementary ways in absorber theory: either the theory is written in completely time-symmetric, half-advanced plus half-retarded, form with no self-interaction, or it is written as a fully retarded interaction with all remote particles plus a self-interaction term which contains a half-retarded minus half-advanced field. The apparent self-force of radiative reaction appearing in (10) is thus seen to be no more than the superposition of the advanced influences of all the other particles in the environment, and thus only *indirectly and approximately* related to the motion of particle a. To emphasize this crucial point of the independence of the force of radiative reaction on the world line of particle a, we note that for an incompletely absorbing universe we would not get the equation of motion (10), and for a "single-particle universe" there would be no radiative reaction whatsoever.

The equation of motion (10) in the nonrelativistic limit becomes

$$m\ddot{\mathbf{x}}=e(\mathbf{E}+\mathbf{v}\times\mathbf{B})+m\tau\dddot{\mathbf{x}}\ , \tag{11}$$

where we introduce the definition $\tau=2e^2/3m$. We will assume for convenience that no magnetic fields are present, and specialize to the case that the external fields acting on particle x can be represented as the sum of a static conservative force and a time-dependent uniform driving force. For each of the Cartesian coordinates, we then have equations of motion

$$m\ddot{x}=-dV/dx+f(t)+m\tau\dddot{x}\ . \tag{12}$$

Now consider a possible single-particle effective Lagrangian for this problem,

$$L=m\dot{x}^2/2-V(x)+[f(t)+m\tau\ddot{x}]x\ . \tag{13}$$

The usual variational principle applied to the action derived from this Lagrangian,

$$\delta S=\delta\int L(x,\dot{x},\ddot{x},\dddot{x},t)dt=0\ , \tag{14}$$

leads to the Euler-Lagrange equations

$$(\partial L/\partial x)-(d/dt)(\partial L/\partial\dot{x})+(d/dt)^2(\partial L/\partial\ddot{x})-(d/dt)^3(\partial L/\partial\dddot{x})=0\ . \tag{15}$$

However, the radiative reaction term completely cancels out of the equations of motion derived in this way. From our discussion following Eq. (10), we see that the radiative reaction term is in reality not intrinsically dependent on the motion of x, since it is merely a convenient approximation, valid under certain conditions, to the influence of the perhaps infinite number of remote absorber particles. We must therefore *not* vary with respect to the radiative reaction term in applying our variational principle. To signify this, we write our Lagrangian as

$$L=m\dot{x}^2/2-V(x)+[f(t)+m\tau\rangle\ddot{x}\langle]x\ , \tag{16}$$

where $\rangle\ddot{x}\langle$ is a special notation which is to indicate that this term represents the net result of summing the influences of all remote particles, which merely happens to be well approximated by the third time derivative of $x(t)$. In the original equation of motion (4), it is clear that there is no explicit self-action. Thus, the forces of radiative reaction are only apparent self-interactions, and should be considered as *functionally independent* of the behavior of the world line of particle a. In a very similar way, and for the same basic reasons, we can also incorporate standard "collisional" damping into our effective Lagrangian by writing

$$L=m\dot{x}^2/2-V(x)+[f(t)+m\tau\rangle\ddot{x}\langle-m\Gamma\rangle\dot{x}\langle]x\ , \tag{17}$$

which leads to an equation of motion

$$m\ddot{x}=-dV/dx+f(t)+m\tau\rangle\ddot{x}\langle-m\Gamma\rangle\dot{x}\langle\ , \tag{18}$$

which can be solved self-consistently by demanding that the solution of the equation of motion coincides with the assumed function $\rangle x\langle(t)$. This we achieve by substituting $x(t)$ for $\rangle x\langle(t)$ in Eq. (18), and solving for $x(t)$. When radiative damping is present, the resulting third-order equation of motion will involve preacceleration phenomena whenever the roots of the associated secular equation have positive imaginary parts as we have discussed previously.[20]

What we have accomplished by this method is a reduction of a conservative many-body (perhaps infinite-

body) problem to a dissipative one-body problem. For radiative reaction this should be an exceedingly good approximation, as the response condition (7) is extremely general. For collisional damping, there is not such a general response condition, and our method may or may not be such a good approximation, depending in detail on the particular system. Note that with only collisional damping, our equation of motion (18) becomes simply the Langevin equation of motion.

If the only thing we achieved by our introduction of the $\rangle\ \langle$ operation was a rederivation of the usual classical equations of motion, we would not have done much. However, with the above single-particle effective Lagrangian it becomes very easy to quantize the classical theory. Since the classical theory involves causality violation, it is then only to be expected that the quantum mechanics we derive will be causality violating also.

PRECAUSAL NONRELATIVISTIC QUANTUM MECHANICS

Of all the existing formulations of quantum mechanics, it is clear that the path-integral approach[22] is best suited to the discussion of causality violating behavior. Indeed, Feynman and Hibbs, in discussing the interaction of a particle with a harmonic oscillator remark ". . . (the action) S contains a product of variables defined at two different times, s and t. The separation of past and future can no longer be made. This happens because the (particle) variable x at some previous time affects the oscillator which, at some later time, reacts back to affect x. No wave function $\psi(x,t)$ can be defined to give the amplitude that the particle is at some particular place x at a particular time t. Such an amplitude would be insufficient for continuing calculations into the future, since at any time one must also know what the oscillator is doing." This particular case of a particle interacting with a harmonic oscillator is very important, since it is the foundation of the description of a particle's interaction with the modes of the quantized electromagnetic field. We will show that in such a case we can still define a wave function, but that it *alone* is not sufficient for continuing calculations into the future.

With our method of incorporating dissipation into the Lagrangian formulation, it is easy to obtain a dissipative, causality-violating, quantum-mechanical theory. From the classical action, the quantum-mechanical propagator can be constructed by the path integral

$$K(x_b,t_b;x_a,t_a)=\int e^{iS[b,a]}\mathcal{D}x\ . \tag{19}$$

The most important paths in the path integral, according to the method of stationary phase, are those for which the action is extremal. The classical path is exactly determined by the condition of stationary action, so that the path-integral approach to quantum mechanics provides a rationale for the standard variational principles of classical mechanics. At this point we make the approximation that in the effective Lagrangian for a specific particle interacting with a remote environment of other particles, all of the terms in the Lagrangian which involve response terms be written in terms of the path of least action, i.e., the classical path. In our specific example of Eq. (17), the radiative damping and collisional damping terms are to be determined by the solutions of the classical equations of motion. In the terminology of Feynman and Hibbs, the influence functional, which represents the averaged response of the environment to the paths of the local particle, is to be approximated by evaluating it in an expansion in terms of time derivatives of the classical path $x_{cl}(t)$. The time evolution of such a quantum system thus depends on the entire course of the path of least action, which is equivalent to the behavior of the corresponding classical system. In conventional quantum mechanics, such a dependence on the future of the time evolution of wave functions would be untenable as it necessarily implies causality violation.

As usual, it is easiest to solve problems in terms of the differential equations obeyed by the states. For this reason, consider the integral equation for the time dependence of a wave function governed by the Lagrangian of Eq. (17),

$$\psi(x,t)=\int\int e^{i\int L[x(t)]dt}\mathcal{D}x(t)\psi(x',t')dx'\ . \tag{20}$$

By considering a time t differing only infinitesimally from t' we discover a Schrödinger-like equation obeyed by ψ,

$$i(\partial\psi/\partial t)=(-1/2m)(\partial^2\psi/\partial x^2)+V(x)\psi -x[f(t)+m\tau\rangle\ddot{x}\langle -m\Gamma\rangle\dot{x}\langle]\psi\ . \tag{21}$$

We thus have a time-dependent, Hermitian, Hamiltonian operator

$$H(t)=(-1/2m)(\partial^2/\partial x^2)+V(x) -x[f(t)+m\tau\rangle\ddot{x}\langle -m\Gamma\rangle\dot{x}\langle]\ . \tag{22}$$

This is a Hamiltonian whose time dependence is determined by the driving force $f(t)$ *and the path of least action* $\rangle x\langle(t)$. Because H is Hermitian, probability is conserved, so that the normalization of states is time independent. This is in contrast to the Gamov procedure of introducing imaginary potentials into the Schrödinger equation to describe dissipation. In the Gamov scheme, the Hamiltonian is not Hermitian, so that energy eigenvalues are imaginary and probability is not conserved. On the other hand, because our Hamiltonian is time dependent, energy is not conserved. If all the (possibly infinite number of) remote particles were kept in our equations, we would find that total energy was conserved, as is discussed by Feynman and Wheeler for the classical case. Obviously, with dissipation of energy to the remote environment, if we only evaluate the energy contained in a given local system, it will not be conserved. The most revolutionary difference between our equation and the usual Schrödinger equation, however, is the dependence on the classical path, which implies that the time evolution of a state is affected by the future as well as the past.

To solve our differential equation, we make use of the general result, valid for any state ψ and operator A,

$$i(d/dt)\langle\psi|A|\psi\rangle=\langle\psi|[A,H]|\psi\rangle+\langle\psi|i\partial A/\partial t|\psi\rangle . \tag{23}$$

Applying this relation to the position and momentum operators in turn, we find Ehrenfest's theorem, and thus find the analogue of Newton's second law of motion,

$$m(d^2/dt^2)\langle x\rangle=\langle -dV/dx\rangle+\langle f(t)\rangle + m\tau(d^3/dt^3)\rangle x\langle -m\Gamma(d/dt)\rangle x\langle . \tag{24}$$

For the Harmonic-oscillator potential, the force $-dV/dx$ is simply proportional to x, so that we solve this problem completely. In the absence of damping, the equation of motion for the expectation value of x is identical to the classical equation of motion. We thus identify

$$\rangle x\langle=\langle x\rangle , \tag{25}$$

and find that the equation of motion for the expectation value of x becomes identical to the classical equation of motion for collisionally and radiatively damped oscillation, viz.,

$$m(d^2/dt^2-\tau d^3/dt^3+\Gamma d/dt+\omega^2)\langle x\rangle=\langle f(t)\rangle . \tag{26}$$

As an explicit example, consider an impulsive driving force, $\langle f(t)\rangle\propto\delta(t)$. Since we have a third-order differential equation, we must specify three boundary conditions to determine the solution. It is most "physical" to let the three required boundary conditions be that $\langle x\rangle$ vanishes as $t\to\pm\infty$, and that $\langle x\rangle$ (and $d\langle x\rangle/dt$) be continuous at $t=0$. This determines $\langle x\rangle(t)$, up to a constant of proportionality, in terms of the three roots $\omega_j, j=1-3$ of the cubic secular equation

$$(-\omega_j^2-i\tau\omega_j^3-i\omega_j\Gamma+\omega^2)\langle x\rangle=0 . \tag{27}$$

Specifically, for the most usual case of undercritical damping there are two roots with negative imaginary parts, and one root with a positive imaginary part, so that

$$\langle x\rangle(t)\propto\begin{cases}(\omega_1-\omega_2)e^{-i\omega_3 t}, & t<0\\ (\omega_3-\omega_2)e^{-i\omega_1 t}+(\omega_1-\omega_3)e^{-i\omega_2 t}, & t>0\end{cases} \tag{28}$$

where

$$\mathrm{Im}(\omega_3)>0 \text{ and } \mathrm{Im}(\omega_{1,2})<0 .$$

Once the time evolution of $\langle x\rangle$ is determined, the quantum-mechanical problem reduces to the form (22) of a time-dependent Hamiltonian, with a driving force well defined as a function of time. In the absence of radiative damping we have only second-order differential equations to solve, and thus the initial wave function contains sufficient information to determine the subsequent time evolution. Once we include radiative damping, this is no longer true. In order to solve for the time evolution of $\langle x\rangle$, we must specify the future conditions that our system will experience. This dependence on the "set up" of the experiment is at the heart of most of the apparent "paradoxes" of conventional quantum mechanics. For example, according to the principle of complementarity, the values that can be measured depend on which variables one sets out to measure. This paradox is trivially resolved if the time evolution of wave functions is explicitly found to depend on the history of the experimental set up. It may appear, on the other extreme, that our present quantum mechanics is based on circular reasoning. Apparently, we cannot solve a problem until we already know the answer. This turns out not to be the case. In general, we find that we must first solve an associated classical mechanics problem, and then plug this solution into the quantum-mechanical wave equation.

To make the above remarks clear, we turn to a specific example which can be exactly solved. By the methods of Feynman and Hibbs the propagator for an impulsively driven, collisionally damped, radiatively damped, harmonic oscillator described by the Lagrangian (17) is exactly

$$K(b,a)=[m\omega/2\pi i\sin(\omega T)]^{1/2}e^{iS_{\mathrm{cl}}[b,a]} , \tag{29}$$

where $T=t_b-t_a$, and

$$\begin{aligned}S_{\mathrm{cl}}=[m\omega/2\sin(\omega T)]\Big\{&\cos(\omega T)(x_b^2+x_a^2)-2x_a x_b+(2x_b/m\omega)\int_{t_a}^{t_b}F(t)\sin[\omega(t-t_a)]dt\\ &+(2x_a/m\omega)\int_{t_a}^{t_b}F(t)\sin[\omega(t_b-t)]dt\\ &-(2/m^2\omega^2)\int_{t_a}^{t_b}dt'F(t')\sin[\omega(t_b-t')]\int_{t_a}^{t'}dt\,F(t)\sin[\omega(t-t_a)]\Big\} ,\end{aligned} \tag{30}$$

where $F(t)=f(t)-m\Gamma(d/dt)\rangle x\langle+m\tau(d^3/dt^3)\rangle x\langle$, and $\rangle x\langle(t)=\langle x\rangle(t)$ is the solution of the classical equations of motion, given by expression (28).

As a specific example of the evolution of wave functions generated by this propagator, consider the initial state

$$\psi(x,0)=e^{-m\omega(x-a)^2/2} , \tag{31}$$

which represents a minimum-uncertainty wave packet whose centroid is located a distance a from the center of the oscillator well. In the absence of damping, i.e., $\Gamma\to0$, the probability distribution evolves in time as

$$|\psi(x,t)|^2=e^{-m\omega[x-a\cos(\omega t)]^2} , \tag{32}$$

so that the centroid exhibits simple harmonic motion,

$$\langle x \rangle(t) = a \cos(\omega t) \ . \tag{33}$$

For a Gaussian initial state, such as (31), our present propagator (30) leaves us with only Gaussian integrals to work out, so that we can exactly solve for $\psi(x,t)$. We find

$$\psi(x,t) = \exp\{ -i\omega t/2 - (m\omega^2/2)[x^2 - x x_0(t) + x_1(t)]\} \ , \tag{34}$$

where $x_0(t), x_1(t)$ are complicated complex functions of time whose explicit form we omit. For the probability distribution we find

$$|\psi(x,t)|^2 = \exp\{ -m\omega[x - \langle x \rangle(t)]^2\} \ , \tag{35}$$

where $\langle x \rangle(t) = 2\,\mathrm{Re} x_0(t)$ is self-consistently given by the classical solution (28).

We have now arrived at the quantum generalization of our previous[20] semiclassical treatment of Compton scattering in the region of the Δ resonance. In the quantum-mechanical case, we now understand that a δ-function driving force

$$f(t) = (1/2\pi) \int e^{-i\omega t} d\omega \ , \tag{36}$$

is simply a superposition of all frequencies with equal amplitude, so that the Fourier resolution of the wave function (34) determines the detailed variation of the cross section of the damped oscillator exposed to any given frequency. Our semiclassical estimate of the energy dependence of the cross section for Compton scattering should be valid as long as the wavelength is not much smaller than the size of the proton, as is approximately true near the Δ resonance. At this point we conclude our discussion of the unconventional aspects of precausal quantum mechanics, and proceed to make contact with conventional quantum physics.

COMPARISON WITH CONVENTIONAL QUANTUM MECHANICS

In any system for which damping can be neglected, all paths in the path integral become independent of the path of least action, and the propagator reduces exactly to the conventional propagator. Even if collisional damping is present, the time evolution of the path of least action is determined by the initial conditions, and so there is no need to specify future conditions. In such cases, the present formulation of quantum mechanics reduces to standard quantum mechanics. A path-integral treatment of such cases has been given by Feynman and Vernon,[23] including extensive discussion of the usual connections between dissipation and fluctuations. For any case in which radiative damping is significant, the linear-dissipation scheme of Feynman and Vernon can only be approximately applied, and our formulation introduces modifications with respect to such conventional treatments.

To see the modifications to conventional quantum mechanics resulting when radiative reaction terms are important in a simple setting, consider the behavior of a two-level problem. That is, suppose that the underlying Hamiltonian is such that only two states are significantly involved. Initially, we suppose that the higher energy level is populated. We then wish to solve for the time dependence of the population of the two states.

Consider exact eigenstates of the time-independent Hamiltonian

$$H_0 = (-1/2m)\partial^2/\partial x^2 + V(x) \ , \tag{37}$$

which satisfy

$$H_0 \psi_k = E_k \psi_k \ . \tag{38}$$

As a specific example one might consider that V represents the potential experienced by an atomic electron. For any eigenstate which satisfies

$$\langle x \rangle = \langle \psi_k | x | \psi_k \rangle = 0 \ , \tag{39}$$

such as is true for any state of definite parity, we also have

$$(d/dt)^n \langle x \rangle \propto \langle \psi_k | [H,x] | \psi_k \rangle = 0 \ . \tag{40}$$

For such a state, we conclude that there are no response terms, as they depend on time derivatives of $\langle x \rangle$. In our framework, a stationary state is truly stationary, and perforce does not decay spontaneously. Our quantum mechanics thus solves the old classical paradox of why atomic electrons do not rapidly spiral into the nucleus as they radiate away their energy. We apparently have a new paradox, in that excited states appear to be stable. In the framework of direct particle-interaction theory,[4,5] however, spontaneous emission is understood as the interaction with remote particles via advanced waves from the future. Spontaneous emission only appears to us macroscopic, "forward-in-time-bound" observers as spontaneous, since we are not able to control the advanced future interactions with remote particles. The best we can do is follow the illusory "quantum" to its ultimate absorption, in order to discover which particle of the future interacted with our apparent spontaneous emitter.

Without specifying when (or whether) an advanced interaction with the future occurs, we are unable to determine the complete time evolution of a state. Assuming that we have an exact eigenstate is equivalent to the assumption that there will never be an interaction with a distant absorber in the future. In a hypothetical one-particle universe, this would be the proper behavior. Since we are in fact living in a complicated universe, it is not reasonable to assume that there will never be an interaction in the future, and we must conclude that *exact* eigenstates are never actually realized. If the decay rate of an excited state is very small, then the incidence of interactions with the future is correspondingly less frequent, and it becomes a good approximation to use an exact eigenstate for the description of the excited state.

Now contrast the picture of electromagnetic radiation in terms of conventional quantum mechanics with the present modified quantum theory. Consider the conventional picture of the spontaneous radiative decay of an excited atomic state, followed by the absorption of the

emitted light by a remote atom. Conventionally, the excited state emits a photon, whose probability distribution spreads out through space in all directions at the speed of light. The leading edge of this probability amplitude distribution is complicated and dependent on the details of the formation process of the excited state. At some point, the photon is absorbed at some remote location, and "instantaneously" the wave function of the photon collapses, so that even locations arbitrarily far removed suddenly have no photon wave function present. In the present picture, on the other hand, an excited state undergoes an action-at-a-distance interaction with a single remote atom, in which a certain amount of energy and momentum are exchanged. The specific atom which interacts is not determined, and the precise time of the interaction is also not determined by the initial conditions. However, once the course of events is known, i.e., which detectors, baffles, sources, etc., are placed where and when, all intermediate amplitudes are determined, and the usual quantum-mechanical calculation of transition probabilities may be made. Precausal quantum mechanics thus lends itself most naturally to the "transactional interpretation" of Cramer.[15]

To make contact with standard quantum-mechanical results we apply the general result (23) to the Hamiltonian operator itself to find

$$d/dt\langle H\rangle=\langle \partial H/\partial t\rangle \ . \tag{41}$$

For simplicity we consider only radiative damping, so that

$$\langle \partial H/\partial t\rangle=-m\tau\langle x\rangle\langle \dddot{x}\rangle \ . \tag{42}$$

For approximately periodic motion, we may integrate by parts twice to find

$$d/dt\langle H\rangle=-\tfrac{2}{3}e^2(d^2/dt^2\langle x\rangle)^2 \ . \tag{43}$$

Note that we have derived the quantum analogue of the Larmor radiation formula without any reference to photons or electromagnetic fields. Our approach to radiative reaction in terms of Ehrenfest relations is similar to that of Ackerhalt and Eberly,[24] except that we do not eliminate third-order time derivatives, as they do, and so we do not eliminate preacceleration phenomena.

We now return to the two-level problem and suppose that we have a state which contains the following superposition of two eigenstates of the time-independent Hamiltonian (37):

$$\psi(x,t)=\alpha(t)\psi_1(x,0)+\beta(t)\psi_2(x,0) \ . \tag{44}$$

The eigenstates $\psi_{1,2}$ have energies E_1,E_2, respectively, so that the total energy of our state is given by

$$\langle\psi|H|\psi\rangle=|\alpha(t)|^2E_1+|\beta(t)|^2E_2 +2m\tau\,\mathrm{Re}[\alpha^*(t)\beta(t)\langle\psi_1|x|\psi_2\rangle\langle\dddot{x}\rangle] \ . \tag{45}$$

The expectation value of x is given by

$$\langle\psi|x|\psi\rangle=2\,\mathrm{Re}(\alpha^*(t)\beta(t)\langle\psi_1|x|\psi_2\rangle) \ , \tag{46}$$

as long as x has no diagonal matrix elements. With this same provision, we find

$$\langle -\partial V/\partial x\rangle=-m\omega_0^2\langle x\rangle \ , \tag{47}$$

where $\omega_0=E_2-E_1$. This may be proven by writing the force operator in terms of the commutator of p and H, and then using the proportionality of p to the commutator of x and H. The quantum-mechanical second law (26) now becomes [with $\langle f(t)\rangle$ representing the interaction which precipitates the decay of the excited state]

$$m(d^2/dt^2-\tau d^3/dt^3+\omega_0^2)\langle x\rangle=\langle f(t)\rangle \ . \tag{48}$$

As a result, we find once again that under very general conditions, the expectation value of $\langle x\rangle$ exhibits damped oscillatory behavior of the form (28), which includes a preacceleration phase. For a system which is initially in the excited state, we have $|\beta(t\to-\infty)|\to1$ and $\alpha(t\to-\infty)\to0$, so that $\langle x\rangle$ vanishes in the remote past. As the time of the interaction approaches, $\langle x\rangle$ increases exponentially with the characteristic preacceleration time constant τ. After the interaction takes place, $\langle x\rangle$ undergoes damped oscillatory behavior, with an approximate frequency given by the energy difference E_2-E_1, and decaying with a rate we will define to be $\Gamma_{\mathrm{rad}}/2$. In the remote future, we find $\beta(t\to+\infty)\to0$ while $|\alpha(t\to+\infty)|\to1$, so that $\langle x\rangle$ vanishes, just as in the remote past. For a narrow-width case, $\Gamma_{\mathrm{rad}}\ll E_2-E_1$, we have, for times t following the initiation of the decay of the excited state, approximately

$$\langle x\rangle\approx x_0e^{-\Gamma_{\mathrm{rad}}t/2}\cos[(E_2-E_1)t+\phi] \ , \tag{49}$$

so that the time derivative which enters the energy-loss equation (42) is

$$(d/dt)^4\langle x\rangle\approx(E_2-E_1)^4\langle x\rangle \ ,$$

and the power loss (42) is

$$P\approx\tfrac{2}{3}e^2(E_2-E_1)^4\langle x\rangle^2 \ , \tag{50}$$

where P is the power radiated. We thus find that the average power radiated decays exponentially. To recover the standard dipole radiation formula, consider a time long enough after the decay is precipitated that $|\alpha|\approx1$, $|\beta|\ll1$, at which point the average power radiated is

$$P_{\mathrm{av}}\to\tfrac{4}{3}e^2(E_1-E_2)^4|\beta|^2|\langle1|x|2\rangle|^2 \ . \tag{51}$$

By normalizing this power emission per unit of excited state probability and per unit of transition energy, we find the usual electric dipole radiation formula for the radiative decay rate,

$$\Gamma_{\mathrm{rad}}=\tfrac{4}{3}e^2(E_2-E_1)^3|\langle\psi_1|x|\psi_2\rangle|^2 \ , \tag{52}$$

completely in agreement with the usual electric dipole emission rate for an atomic transition from state ψ_2 to state ψ_1. Our derivation contrasts with that of conventional quantum mechanics, in that we have no need to assume *a priori* that we have "adiabatic switching on" of a radiative reaction perturbation in the remote past. Our formalism automatically includes the transition

from the remote past through the present to the remote future.

We may also derive the transition rates for stimulated emission or absorption of radiation from our Hamiltonian of Eq. (22) by including the time-dependent driving term

$$\mathbf{f}(t)=e\mathbf{E}(t)\ , \tag{53}$$

where $\mathbf{E}$ is the external electric field. If we suppose that the electric field is turned on at $t=0$, and that initially $\alpha=1$, $\beta=0$, then the quantity $\langle \mathbf{x} \rangle$ initially vanishes. For a short time afterwards, the radiation-reaction term will be negligible compared to the external driving force term of Eq. (53). We many therefore write our Hamiltonian as

$$H(t)=H_0-e\mathbf{x}\cdot\mathbf{E}(t)\ . \tag{54}$$

For this problem we may use Dirac's conventional time-dependent perturbation theory to solve for the time evolution of the amplitudes $\alpha(t)$ and $\beta(t)$. The result is that we find the conventional expressions for the stimulated emission or absorption probabilities, leading to the total transition probabilities W,

$$\begin{aligned} W_{1\leftarrow 2}&=A_{12}+B_{12}U(\omega)\ , \\ W_{2\leftarrow 1}&=B_{21}U(\omega)\ , \end{aligned} \tag{55}$$

where $U(\omega)d\omega$ is the energy density of the electromagnetic field in the angular frequency interval $(\omega,\omega+d\omega)$, and

$$B_{12}=B_{21}=(4\pi^2e^2/3)\,|\,\langle 1\,|\,x\,|\,2\rangle\,|^2\ , \tag{56}$$

is the Einstein coefficient for stimulated emission or absorption, while

$$A_{12}=(4e^2\omega^3/3)\,|\,\langle 1\,|\,x\,|\,2\rangle\,|^2\ , \tag{57}$$

is the Einstein coefficient for spontaneous emission, which we have already derived in Eq. (52) above. It is well known that these Einstein coefficients lead, under conditions of thermodynamic equilibrium, to the Planck black-body spectrum

$$U(\omega)=(\omega^3/\pi^2)[\,\exp(\omega/kT)-1\,]^{-1}\ . \tag{58}$$

The spectral density can be written as

$$U(\omega)=\omega^3 n(\omega)/\pi^2\ , \tag{59}$$

where

$$n(\omega)=1/[\,\exp(\omega/kT)-1\,]\ , \tag{60}$$

is the number of "photons" of angular frequency ω. In our theory, the number of photons is equivalent to the number of emitter-absorber interactions "in progress."

As a specific example, consider a 1 g mass of hydrogen gas in a 1 liter container whose walls are assumed perfectly reflecting. Suppose there is precisely one atom in its $2p$ state initially, while all the others are in their ground state, corresponding to an equilibrium temperature of 2162 K. Approximately once every nanosecond the excited atom emits its quantum of energy and returns to the ground state. In a very short time the quantum is absorbed by some other atom, and it becomes excited to the $2p$ state. Most of the time the state of the entire system consists of a single excited atom with no radiation field at the 10.2 eV energy. During the course of an emitter-absorber interaction it is impossible to attribute the quantum to either the emitter or the absorber. The radiation field acts as a "stand in" to represent the "quantum in transit" from the emitter to the absorber. While the quantum of energy is in transit, we may say that the radiation field contains a single 10.2-eV photon. The strength of the adjunct electromagnetic field at the position of any given atom must be of the appropriate intensity to account for the time average absorption rate. If all atoms are in their ground state, then there is no possibility of one atom exciting another, and the entire system remains in this static situation. The total energy density of the system is merely the mass energy density of the ground-state atoms, and is finite.

In conventional quantum electrodynamics, on the other hand, the modes of the electromagnetic field are independent degrees of freedom, and therefore possess a zero-point energy. As a result, the vacuum spectral energy density of the electromagnetic field is

$$U_{\text{conv}}(\omega)=(\omega^3/\pi^2)\{\tfrac{1}{2}+1/[\,\exp(\omega/kT)-1\,]\}\ . \tag{61}$$

On integrating over all frequencies, the zero-point contribution to the spectrum diverges,

$$\int_0^\Omega d\omega\,\omega^3/2\pi^2=\Omega^4/8\pi^2\rightarrow\infty\quad(\text{as }\Omega\rightarrow\infty)\ . \tag{62}$$

Conventionally this vacuum-energy divergence is eliminated by subtracting an infinite constant from the Hamiltonian.

It is conventionally argued that there is direct experimental evidence for the zero-point fluctuations in that effects such as the Lamb shift, or the Casimir effect would not be present without the zero-point fluctuations in the electromagnetic field. Since the Lamb shift is well established experimentally, it is widely believed that zero-point fluctuations are "real," and that they require the electromagnetic field to have its own independent degrees of freedom.

On the other hand, a few authors[25–27] have pointed out that the effects of the zero-point fluctuations are just the same as the effects produced by the forces of radiation reaction. We quote Jaynes,[25] "This complete interchangeability of source-field effects and vacuum-fluctuation effects does not show that vacuum fluctuations are 'real.' It shows that source-field effects are the same *as if* vacuum fluctuations were present. The radiating atom is indeed interacting with an electromagnetic field of the intensity predicted by the zero-point energy, but this is just the atoms own radiation reaction field."

We can see this very directly and quickly from the expression for the emission probability in Eq. (55). In the absence of external fields, the emission probability is given by the Einstein A coefficient. If we rewrite this in terms of a B coefficient times an "equivalent" radiation field $U(\omega)=\omega^3/\pi^2$, we would get exactly the same emis-

sion rate. But this equivalent radiation field is exactly that which corresponds to an occupation number $n(\omega)=1$. We may say that the photon responsible for the field is just the photon being emitted.

Again quoting Jaynes: "(The divergences of the vacuum fluctuations) disappear as soon as we realize that, in order to account for spontaneous emission, there is no need for this energy density to be present in all space, at all times, in all frequency bands. It is produced automatically by the radiating atom, but in a more economical way; only the field component that is needed, where it is needed, when it is needed, and in the frequency band needed."

In our Hamiltonian, Eq. (22), the force of radiation reaction is not present as long as $\langle \mathbf{x} \rangle = 0$. It is only while an emitter-absorber interaction is taking place that $d^3/dt^3 \langle \mathbf{x} \rangle \neq 0$, and that the radiation reaction term is present. Since this produces a contribution to $H(t)$ which cannot be distinguished from the equivalent zero-point fluctuating field contribution, we will still have the consequences of the zero-point fluctuations without the associated divergences. We may therefore follow the arguments of Welton's analysis of the Lamb shift.

Welton[28] considers the effects on an electron produced by the equivalent zero-point fields. Because of the fluctuations, the potential "seen" by an electron becomes smeared out,

$$[V(\mathbf{x}+\Delta\mathbf{x})]_{\text{av}} \approx (1+\tfrac{1}{6}[\Delta\mathbf{x}^2]_{\text{av}}\nabla^2)V(\mathbf{x}) \ . \tag{63}$$

For the Coulomb potential, the Laplacian produces a δ function at the origin, so that only s states are affected. Welton estimates the mean square fluctuation of the coordinate by considering the equation of motion for a free electron,

$$m(d^2/dt^2)\mathbf{x} = e\mathbf{E}_{\text{ZP}} \ . \tag{64}$$

For a single-frequency component we have

$$\mathbf{x}(\omega) = -e\mathbf{E}(\omega)/m\omega^2 \ , \tag{65}$$

so that

$$[\mathbf{x}^2]_{\text{av}} = (e^2/m^2)\int_0^\infty \mathbf{E}^2_{\text{ZP}}(\omega)d\omega/\omega^4 \tag{66}$$

$$= (2e^2/\pi m^2)\int_0^\infty d\omega/\omega \ . \tag{67}$$

This integral diverges logarithmically in Welton's treatment. He must argue that both the high-energy and low-energy bounds should in fact be finite values, and therefore a finite integral results. If we had instead used a somewhat more realistic equation of motion

$$m(d^2/dt^2 - \tau d^3/dt^3 + \omega_0^2)\mathbf{x} = e\mathbf{E} \tag{68}$$

then we would have found a mean square fluctuation of

$$[\mathbf{x}^2]_{\text{av}} = (e^2/m^2)\int \mathbf{E}^2_{\text{ZP}}(\omega)\,|\,\omega^2 + i\omega^3\tau - \omega_0^2\,|^{-2}d\omega \ , \tag{69}$$

which is completely finite, and approximately equal to

$$(2e^2/\pi m^2)\int_{\omega_0}^{1/\tau} d\omega/\omega \ . \tag{70}$$

Our treatment is still unsatisfactory, since the upper cutoff of the integral should more realistically be at the mass m, since relativistic effects will be important at that point. Also, the lower cutoff will not be adequately described by the oscillator approximation for the effects of binding which we have introduced in Eq. (68). Our main point is that there indeed *are* significant effects produced by the radiation-reaction term and that, with a more careful treatment, they should reproduce the results of the conventional nonrelativistic treatment. That vacuum fluctuations and radiation reaction can be interchanged in this way is a result of a profound fluctuation-dissipation theorem, as pointed out by Jaynes,[25] Milonni,[26] and others.

LIMITATIONS AND OMISSIONS

We have purposely omitted treatment of relativistic effects. We have also not discussed the effects of spin. We could have treated spin by introducing it into the Hamiltonian of Eq. (22), with the usual Pauli coupling to the magnetic field and with a radiative-reaction-force term proportional to the third time derivative of the expectation value of the magnetic moment operator, in analogy with the treatment of the electric radiative-reaction term. We would then be able to treat magnetic dipole radiation in the same way as we have treated electric dipole radiation.

At the level of the approximations we have made, we do not have quadrupole, or higher multipole radiation. This is the quantum analogue of the fact that nonrelativistic Larmor radiation is only dipole radiation. If we had kept relativistic corrections in the original equations of motion, we would also end up with higher multipole radiation terms here, and increasing powers of (v/c) would correspond to increasing multipole order. Even without explicitly calculating such a multipole expansion, we expect that as a general rule to account for precausal behavior in expressions derived conventionally, we should replace all time dependence of the form $e^{-i\omega t-\Gamma t/2}$ in conventional expressions by a time dependence of the general form of Eq. (28). For most cases in atomic physics, the resulting change will introduce only very minute modifications into the corresponding energy spectrum. Only in cases where the precausal time constant τ is significant compared to the characteristic frequency of a system will there be noticeable differences.

The relevant parameter determining the relative importance of the deviations between our theory and conventional quantum mechanics is set by $\omega\tau$, where ω is an energy scale characteristic of the given system and τ is the preacceleration time constant. For any processes involving nonrelativistic electrons, $\omega < m_e$, so that $\omega\tau \ll 1$. On the other hand, for high-energy processes involving electrons, relativistic quantum-mechanical effects greatly dominate the essentially classical preacceleration effects. Thus, as we have discussed earlier,[19] it is extremely difficult to see any precausal effects in the physics of systems dominated by electrons. For the proton, however, where apparently $m_p\tau_p \approx 1$, we do expect to see effects at the 10% level for $\omega \approx 300$ MeV. Thus we are drawn almost uniquely to the $\Delta(1232)$ resonance in proton Compton scattering as essentially the only quantum sys-

tem and energy scale for which our nonrelativistic approach differs in a readily measurable way from that of conventional quantum mechanics.

SUMMARY

Motivated by the expression of the force of radiative reaction in terms of the influence of the environment rather than an intrinsic self-action, we wrote an effective time-dependent Lagrangian which was able to account for dissipation. We quantized the action generated by this Lagrangian via the path-integral approach. The resulting generalization of quantum mechanics appears to be able to account for a wider variety of phenomena than conventional quantum mechanics. We also notice a deeper connection between our quantum mechanics and classical mechanics than is apparent with conventional quantum mechanics, since we find that the classical behavior of a system plays a key role in determining the detailed behavior of the quantum system. We solve exactly for the behavior of a simple damped harmonic oscillator, and approximately the behavior of a radiatively decaying two-level system. We point out that our precausal quantum mechanics reduces to conventional quantum mechanics in all cases for which the characteristic precausal time scale τ is negligible.

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